Dynamics

A Lecturebook

Bob Lyon

University of Kansas Libraries

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Front and back cover image: SpaceX's Falcon 9 rocket and Dragon spacecraft launched from Launch Complex 40 at the Cape Canaveral Air Force Station, Florida, for their fourth official Commercial Resupply (CRS) mission to the orbiting lab on Sunday, September 21 at 1:52am EDT. Dragon returned to Earth with a parachute-assisted splashdown off the coast of southern California on October 25. Dragon is the only operational spacecraft capable of returning a significant amount of supplies back to Earth, including experiments. This image is in the Public Domain.

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Preface

Engineering mechanics deals with the effects on bodies that are subjected to forces. The two primary subdivisions of mechanics are statics; where the bodies remain at rest, and dynamics; where the bodies are put in motion.

The aim of this dynamics text is 1) to develop an understanding of the basic principles governing the response of bodies to forces, 2) to develop an ability to solve problems simply and logically, and 3) to apply these basic principles to practical engineering problems.

This text grew out of lecture notes used for a sophomore level active learning classroom. It is intended as a concise, straightforward representation of fundamental dynamics concepts. Each module contains an explanation of a fundamental principle, followed by example problems, group activity problems, and homework problems. The example problems and group activity problems contain plenty of white space, to enable students to work the problems in the Lecturebook, either in or out of an active classroom setting.

Topics are organized to enhance understanding. After introductory material, the text is divided into four main groupings that discuss in turn the kinematics and kinetics of particles, and then the kinematics and kinetics of rigid bodies. The content of the book can be covered in a three semester hour course.

Acknowledgments

I am grateful to all those who have supported and assisted me in carrying out this project. I would like to thank the University of Kansas' Libraries for the Open Educational Resources Grant Initiative, which funded this work.

I am very grateful to Josh Bolick and Rebecca Orozco from the Shulenburger Office of Scholarly Communication & Copyright. I would like to thank the University of Kansas Department of Civil, Environmental and Architectural Engineering, and particularly all the CE 301 undergraduate students and GTA's that have contributed to the development of this work.

Introduction

Lesson 1

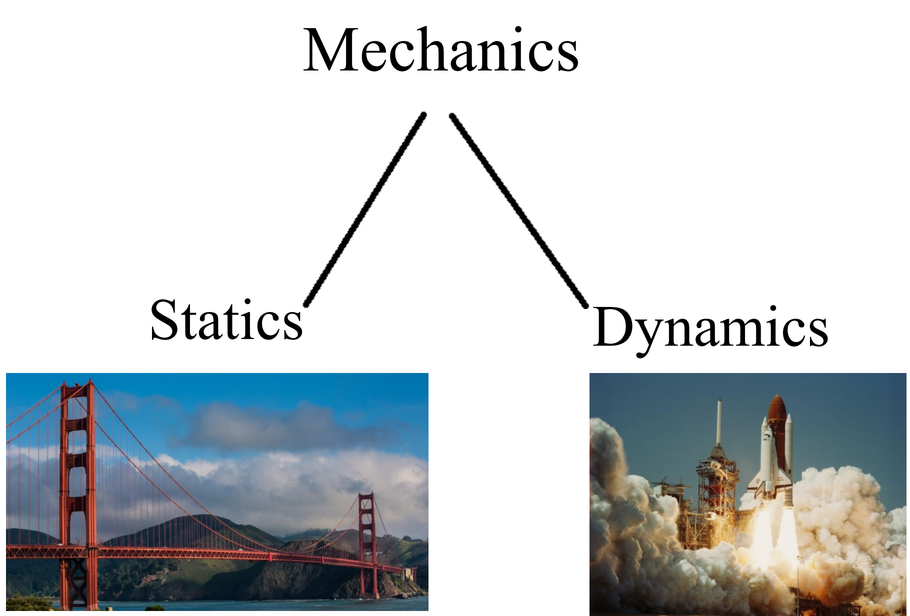
1. Mechanics: The study of how bodies react to forces acting on them.

Statics: The study of bodies in equilibrium.

Dynamics: The study of bodies in motion.

2. Let’s introduce our study of dynamics. Up until now we have covered the concept of statics, where the rigid body is in equilibrium and the sum of the forces and moments are zero. When a body is in motion these equations are no longer zero and the realm of mechanics is known as dynamics.

3. Dynamics has its foundation in **Newton’s Laws of Motion**. We will define the parameters that are used to describe the motion of a body, i.e. position, velocity and acceleration.



Golden Gate Bridge

Space Shu le Launch

4. **Newton's Laws of Motion**

Newton's First Law - A particle at rest will remain at rest….**unless** the particle is acted on by an **unbalanced force**.

Newton's Second Law - When a particle is acted on by an unbalanced force, the particle will be accelerated in the direction of the force. The magnitude of acceleration will be proportional to the force and inversely proportional to the mass of the particle. F = ma

Newton's Third Law - For every action, there is an equal and opposite reaction.

5. Two sub-categories of dynamics

Kinematics - study of an object's movement

Kinetics - study of the **relationship** between the **motion** and the **forces** that cause the motion.

6. We may consider three classes of motion, motion along a straight line, motion in a plane along a curved path, and motion in space along a curved path.

Rectilinear motion - motion along a straight line.

Plane curvilinear motion - motion in two dimensions along a curved path.

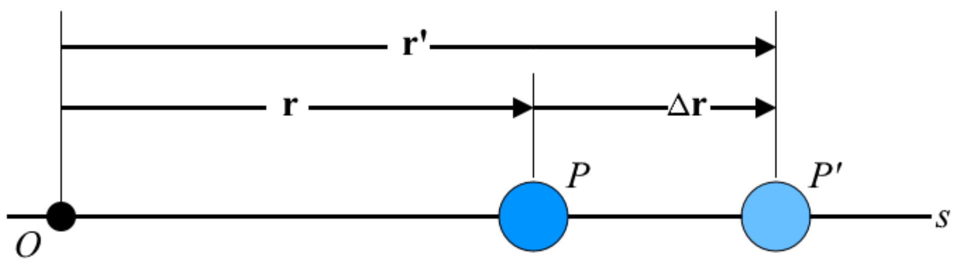
General curvilinear motion - motion in three dimensions along a curved path.

Rectilinear Motion

Lesson 2

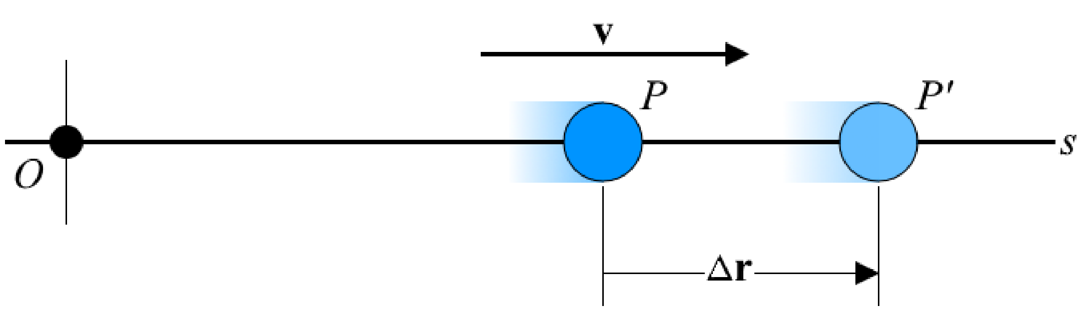
1. For a particle in rectilinear motion, its position is specified by a scalar distance from a fixed origin at any instant of time.

We define **displacement** as the **change in position** over a given time interval. The displacement depends only upon the starting and finishing position of the particle, not the path that was taken to get there. The **total distance traveled** represents the total length of the path over which the particle travels.



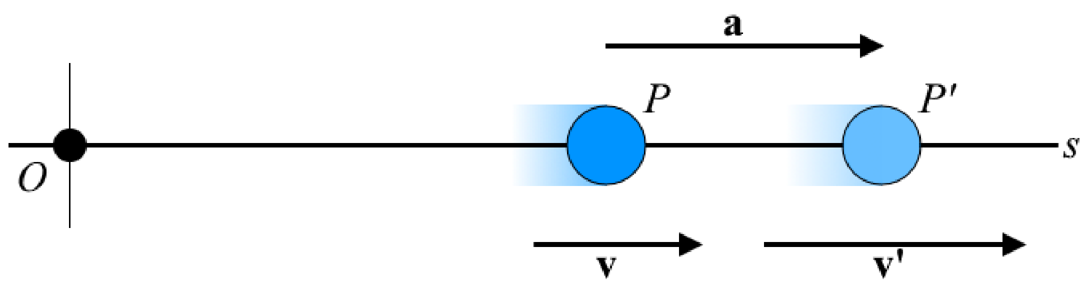
Displacement

Velocity measures how fast the position of the particle is changing. The average velocity is the displacement divided by the time interval. Shrinking the time interval down to zero defines the instantaneous velocity at any point as **dr/dt**.



Velocity

Acceleration measures how fast the velocity of the particle is changing. The instantaneous acceleration is **dv/dt**, or $\mathbf{d^2r/dt^2}$.



Acceleration

2. The kinematic relationship for rectilinear motion may be summarized as shown. Differentiate position with respect to time to get velocity, and differentiate velocity to get acceleration. Integrate acceleration to get velocity, and integrate velocity to get position.

- Differentiate position to get velocity and acceleration.

$$v = ds/dt \; ; \qquad a = dv/dt \quad \text{or} \quad a = v\, dv/ds$$

- Integrate acceleration for velocity and position.

Velocity: Position:

$$\int_{v_o}^{v} dv = \int_{o}^{t} \mathrm{a}\, dt \;\text{ or }\; \int_{v_o}^{v} v\, dv = \int_{s_o}^{s} \mathrm{a}\, ds \qquad \int_{s_o}^{s} ds = \int_{o}^{t} v\, dt$$

- Note that s_o and v_o represent the initial position and velocity of the particle at t = 0.

3. These relationships take a specific and simplified version, for cases where the acceleration is constant. These **three constant acceleration equations** are:

$$\mathbf{v = v_0 + a_c t}$$
$$\mathbf{s = s_0 + v_0 t + (1/2) a_c t^2}$$
$$\mathbf{v^2 = v_0^2 + 2a_c(s - s_0)}$$

Remember that these familiar equations may only be applied for constant acceleration conditions.

Example 1

Given: A single particle moves across a straight line in the positive direction with velocity of $v = 6t+9t^2$ in meters per second.

Find: The position and acceleration of the particle at $t = 7$ seconds.

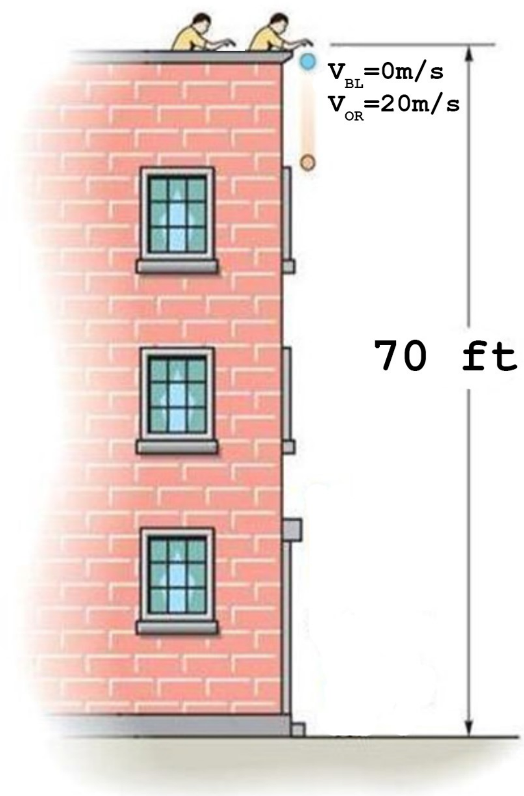
Plan: The velocity is given as a function of time. Take the integral of the velocity function to find the position function. To find acceleration take the derivative of velocity function, Plug in 7 for t for all three equations.

Example 2

Given: Jordan and John stand on top of a 70 ft building. They release two identical balls at the same time from the same height. John releases a blue ball with an initial velocity of V_{BL}= 0 m/s while Jordan throws an orange ball down with an initial velocity of V_{OR} = 20m/s.

Find: The time difference between each ball's impact and their respective final velocities.

Plan:



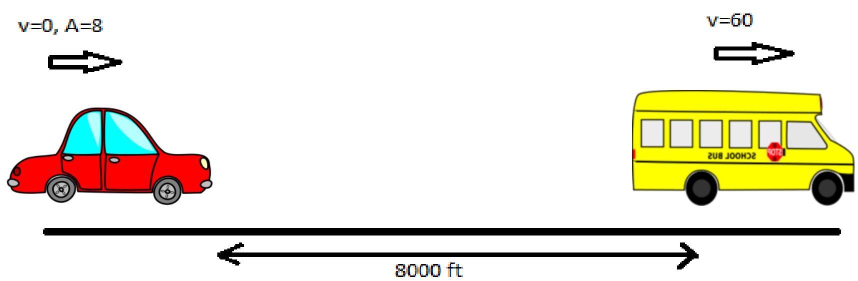
Homework Assignment # 1

1. A man starts at home ($x=0$) and, over about 30 seconds, he accelerates towards a steady state speed of 4 m/s according to the function: $v(t) = 4(1-e^{-t/(30s)})$ m/s and his whole ride lasts 1000 seconds (about 17 minutes). (a) How far did he travel? (Hint: solve the differential equation) (b) What would the distance covered by the man be if the whole trip was traveled at a steady state speed of 4 m/s? (c) How less/more has the man traveled in part (a) as compared to part (b)?

2. A 0.5kg mass starts from rest and attains a speed of 20 m/s $\hat{\mathbf{i}}$ in 4 s. Find the constant acceleration of the mass.

Lesson 2 Group Work

A car starts from rest at $t = 0$ and travels along a straight road with a constant acceleration of 8 ft/s^2 until it reaches a speed of 120 ft/s. Afterwards it maintains this speed. Meanwhile ($t = 0$), a bus moves slowly down the same road 8000 ft ahead in the same direction. The bus moves at a velocity of 60 ft/s with no acceleration. Determine the distance traveled by the car when it passes the bus.



Determine the time for the car to achieve $v = 120$ ft/s

Determine the distance the car travels during this time. (s_1)

Write the equation for the distance the bus travels at constant velocity 60 ft/s. (s_3)

Write the equation for the distance the car travels after that at constant 120 ft/s velocity. (s_2)

Set $s_1 + s_2 = 8000 + s_3$ ft and solve for t.

Homework Assignment #2

1. As a train accelerates uniformly it passes successive kilometer marks while traveling at velocities of 2m/s and then 10 m/s. Determine the train's velocity when it passes the next kilometer mark and the time it takes to travel the 2-km distance.

2. The acceleration of a particle traveling along a straight line is $a = (8 - 2s)\ m/s^2$, where s is in meters. If $v = 0$ at $s = 0$, determine the velocity of the particle at $s = 2m$, and the position of the particle when the velocity is maximum.

3. From the data collected in class today for each of the two students walking:

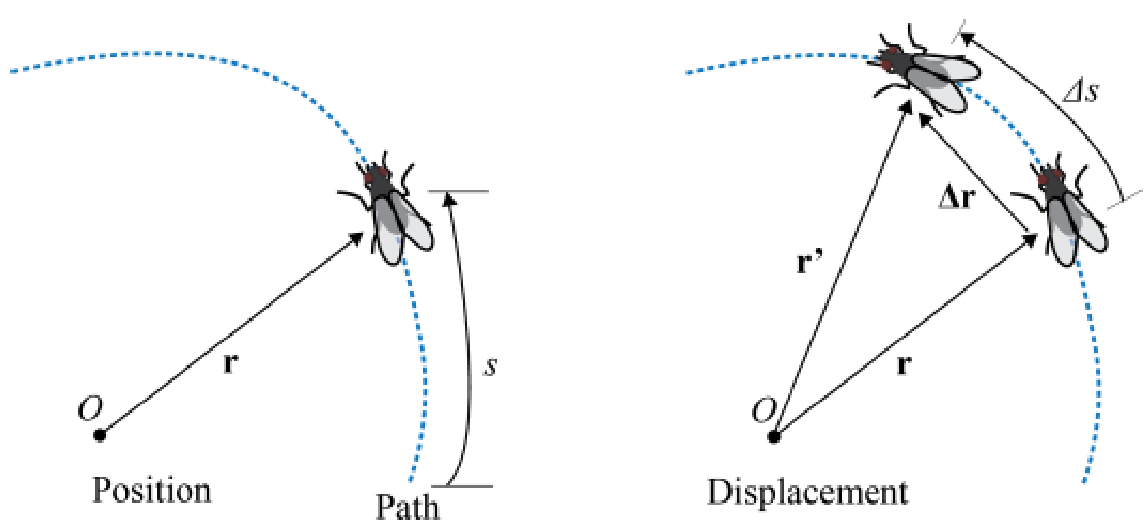
- Calculate the average velocity for each individual segment
- Calculate the overall average velocity
- Determine the average acceleration between segments 1-2, 2-3 and 3-4
- Determine the overall average acceleration
- Create a velocity versus time graph. Draw a horizontal line on the graph that represents overall average velocity
- Create an acceleration versus time graph. Draw a horizontal line on the graph that represents overall average acceleration
- List all assumptions that were applied in your calculations Discuss results and list possible sources of error

Curvilinear Motion

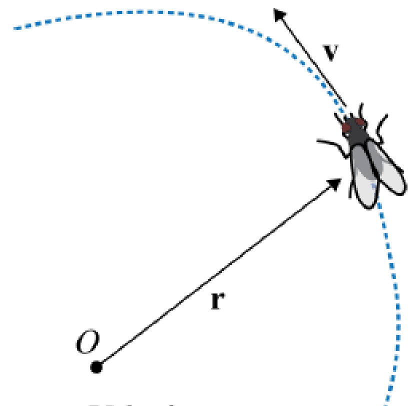
Lesson 3

1. Two-dimensional curvilinear motion occurs when a particle moves along a curved path that lies in a plane. Rectilinear motion in the previous lesson was able to be described with scalars. Curvilinear motion will need to be described by vectors because the position, velocity, and acceleration all have both magnitude and direction. There are three different coordinate systems that are used to describe curvilinear motion depending of the given information of the problem. The first coordinate system we will look at is **rectangular Cartesian coordinates**.

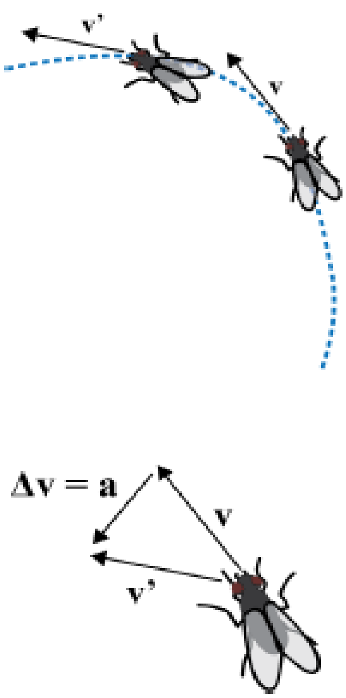
2. The position of the particle is described by a position vector extending from a fixed origin to the location of the particle. The position vector has both magnitude and direction and is a function of time. We define the displacement of a particle as the difference between position vectors at the beginning and end of a discrete time interval.



3. Velocity describes how fast the position of the particle is changing. The direction of the velocity vector is **<u>always tangent</u>** to the path of motion. The magnitude of the velocity vector is referred to as the speed. We can define the average velocity over a time interval as $\mathbf{\Delta r}/\Delta t$. Taking the limit as Δt approaches zero gives us the definition of instantaneous velocity as $d\mathbf{r}/dt$.



Velocity



Acceleration

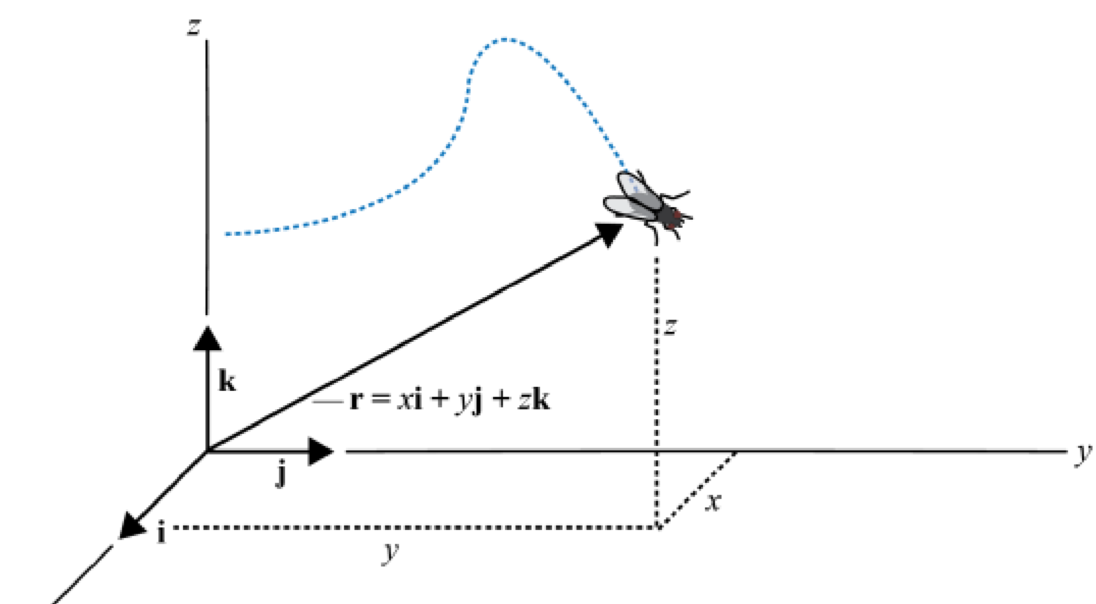
4. Acceleration is how fast the velocity of the particle is changing. The direction of the acceleration vector is usually <u>not</u> tangent to the path. We can define the average acceleration over a time interval as $\mathbf{\Delta v}/\Delta t$. Taking the limit as Δt approaches zero gives us the definition of instantaneous acceleration as $d\mathbf{v}/dt$ or $d^2\mathbf{r}/dt^2$.

For general reference:

- Differentiate to find: Position→Velocity→Acceleration
- Integrate to find: Acceleration→Velocity→Position

5. Let's particularize these general relationships with respect to a rectangular coordinate system. The position vector may be expressed in terms of its x, y and z components. Each of these components may be functions of time, and expressed as a magnitude times the unit vectors **i**, **j** and **k**.

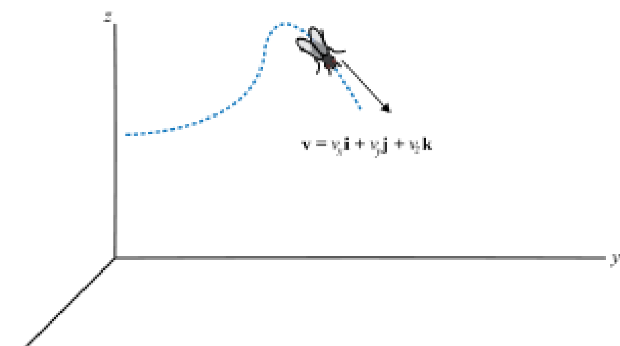
- Particle's position vector: $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
- Magnitude of the position vector is the square root of the sum of the squares of each component: $r = (x^2 + y^2 + z^2)^{0.5}$
- Unit vector in the direction of the position vector is the position vector divided by its magnitude: $\boldsymbol{u_r} = (1/r)\boldsymbol{r}$



Position

6. We obtain the velocity vector representation in rectangular co-ordinates by differentiating the position vector.

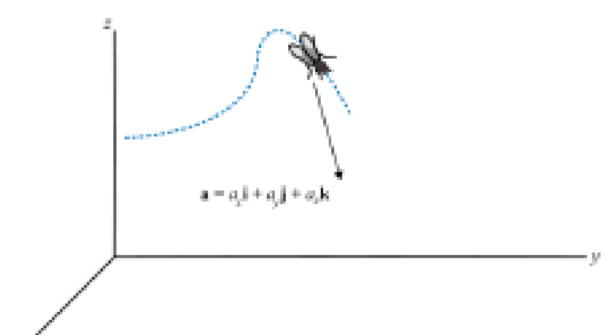
- Velocity vector of a particle: $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$
- v_x, v_y, and v_z may also be written as $\dot{x}$, $\dot{y}$, and $\dot{z}$ respectively.
- Magnitude of velocity vector is the square root of the sum of the squares of the components: $v = (v_x^2 + v_y^2 + v_z^2)^{0.5}$



Velocity

7. We obtain the acceleration vector representation in rectangular coordinates by differentiating the velocity vector.

- Acceleration vector of a particle: $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$
- a_x, a_y, and a_z may also be written as $\ddot{x}$, $\ddot{y}$, and $\dot{z}$ respectively.
- Magnitude of acceleration vector is the square root of the sum of the squares of the components: $a = (a_x^2 + a_y^2 + a_z^2)^{0.5}$



Acceleration

Example 1

Given: The motion of two particles (A and B) is described by the position vectors

$\boldsymbol{r_A} = [3t\,\boldsymbol{i} + 9t(2 - t)\,\boldsymbol{j}]$ m and

$\boldsymbol{r_B} = [3(t^2 - 2t + 2)\,\boldsymbol{i} + 3(t - 2)\,\boldsymbol{j}]$ m.

Find: The point at which the particles collide and their speeds just before collision.

Plan: 1) The particles will collide when their position vectors are equal, or $\boldsymbol{r_A} = \boldsymbol{r_B}$.

2) Their speeds can be determined by differentiating the position vectors.

Example 2

Given: The acceleration of a particle is given as

$$\boldsymbol{a} = [\ 0.7\ t^3\ \boldsymbol{i} + 4\ t^2 \boldsymbol{j} + (16t)\ \boldsymbol{k}]\ m/s^2.$$

When t = 0, x = y = z = 0.

Find: The magnitude of the velocity and position of the particle when t = 4s.

Plan: 1) Determine the velocity and position vectors by integrating **a**, then integrating **v.**

2) Determine the magnitude of the velocity and position vectors at t = 4 s.

Lesson 3 Group Work [1]

1.The position of a particle is defined by $r = 1/3t^3\ \boldsymbol{i} + (1/6t^3 + 0.5t^2)\ \boldsymbol{j}$], find the magnitude of acceleration at t = 2 seconds.

2. $V_A = t^2\ \boldsymbol{i} + 4t\ \boldsymbol{j}$ and $V_B = (t+2)\ \boldsymbol{i} + t^3\ \boldsymbol{j}$, at what time are the velocities of A and B equal.

3. A particle moves from A: (3,2) to B: (9,-7) in 3 seconds. What is the average velocity of the particle?

4. A particle has the following velocity components: $v_x = 4t$ and $v_y = 3t^2$. Find the position function at t = 4 if x = y = 0 when t = 0.

Lesson 3 Group Work [2]

Given: The curvilinear motion of a particle has position $x = 27 - t^3$ and acceleration $a_y = 6t - 6$ where x is in meters and a_y is in m/s^2. Additionally, $v_y = 0$ and $y = 25$ when $t = 0$.

Find: Plot the first three seconds of the path and find each component's velocity and acceleration at t = 0,1,2, and 3.

Plan: Write the expression for the x-component of the velocity vector, and the x-component of the acceleration vector.

Write the expression for the y-component of the velocity vector, and the y-component of the position vector.

Calculate x, y, V_x, V_y, a_x, and a_y when t = 0,1,2, and 3. Then plot the path.

Homework Assignment # 3

1. A particle which moves with curvilinear motion has coordinates in millimeters which vary with time t in seconds according to $x = 3t^2 - 4t$ and $y = 4t^2 - 1/3t^3$. Determine the magnitude of the velocity v and acceleration a and the angles which these vectors make with the x – axis when $t = 2$ s.

2. A car accelerates for 12 seconds by the expressions $a_x = 3t - 4$ and $a_y = 2t^2 + 3$ in m/s^2. After 12 seconds, the car travels at a constant speed. Determine the constant speed the car travels after the initial 12 second period, and find the position of the car after 50 seconds.

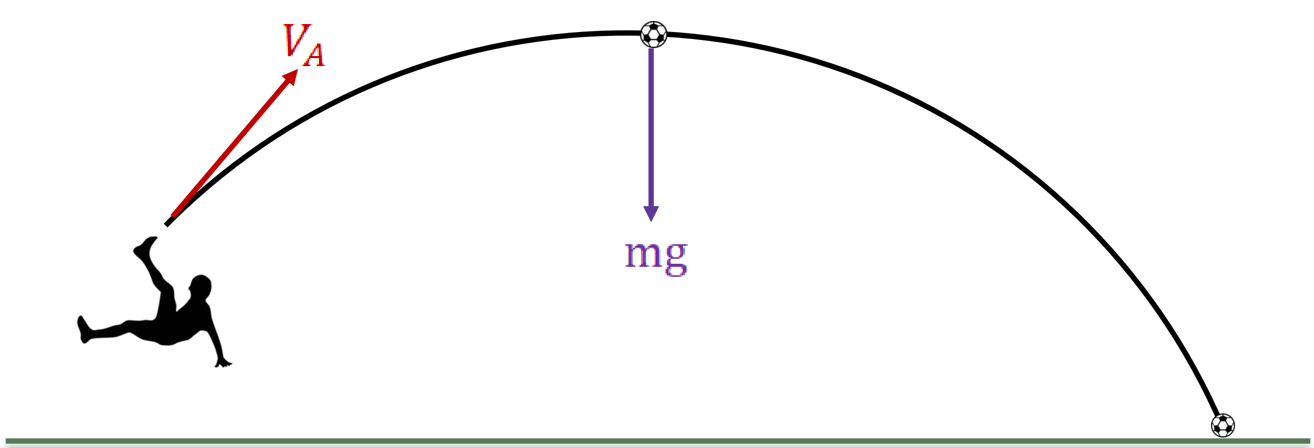
Projectile Motion

Lesson 4

1. **Projectile motion** is a particular case of curvilinear motion that lends itself very well to using rectangular coordinates.

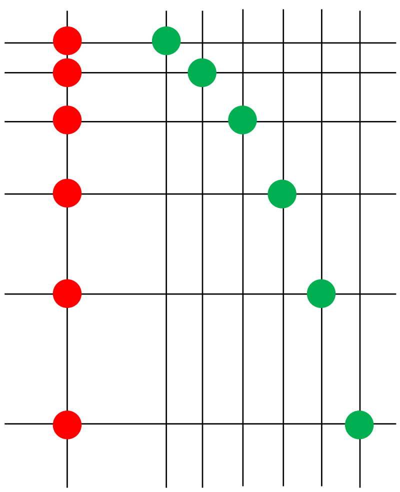
2. A projectile is an object modelled as a particle with an initial velocity that moves through the air unrestricted. There are only two forces acting on the particle: the object's **weight** and **air resistance**. However, air resistance may often be considered negligible, leaving only the weight acting on the object.

The figure below shows a soccer ball in projectile motion. The ball has an initial velocity when kicked, and the only acting force is its weight acting vertically downward.



Soccer ball in projectile motion

3. Since the only force is the object's weight in the vertical direction, projectile motion has **independent** horizontal and vertical motion. This allows us to treat a curvilinear problem as two rectilinear problems. The vertical motion has a **constant acceleration** downward due to gravity. Horizontal motion has zero acceleration, and therefore a **constant velocity**, which equals the horizontal component of the initial velocity. With these conditions, We can apply the kinematic equations for rectilinear motion in both the horizontal and vertical directions.



Falling ball illustration

Consider the two balls on the left. The red ball falls from rest, whereas the green ball is given a horizontal velocity. Both balls have a constant downward acceleration.

Each dot represents the ball's position at equal time intervals. Notice how both balls remain at the same *elevation* at any instant, a result of their equal acceleration. Also notice that the *horizontal distance* between successive positions of the green ball is constant. This occurs because the velocity in the horizontal direction remains constant.

4. Since the acceleration is zero in the horizontal direction, we can substitute $\mathbf{a_x = 0}$ into the constant acceleration position equation previously derived, and we obtain the **kinematic equation** governing the **horizontal motion** of a projectile:

$$\mathbf{x = x_0 + v_{0x}t}$$

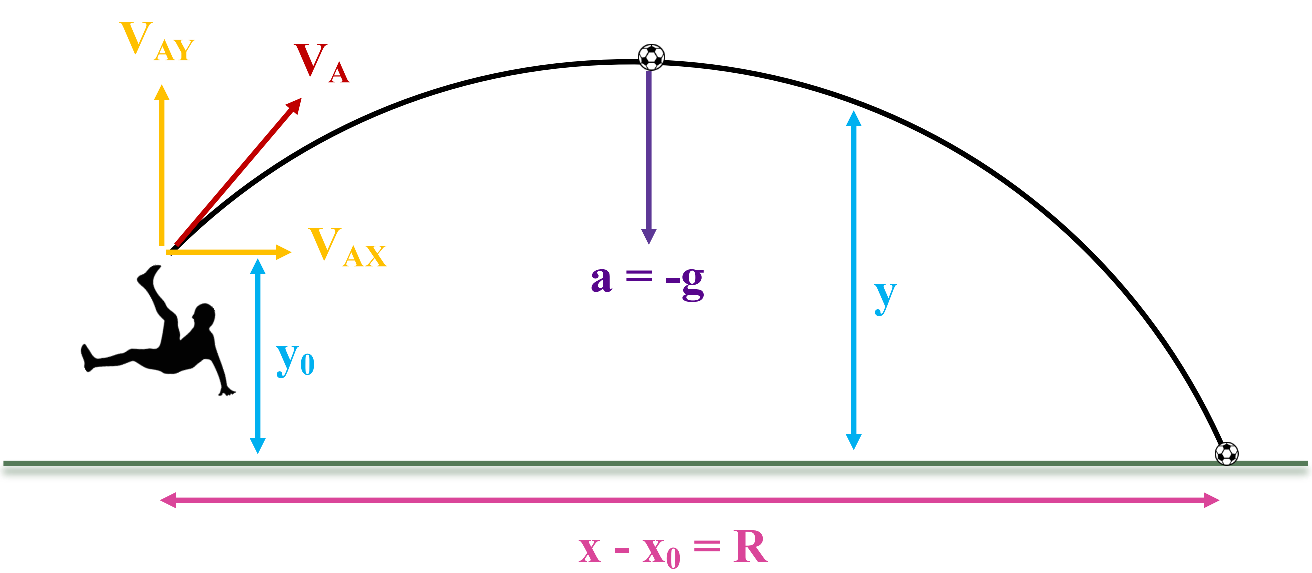
5. Since the object's downward acceleration is due to gravity, we can substitute $\mathbf{a_y = -g}$ into the constant acceleration equations previously derived, and we obtain the **kinematic equations** governing the **vertical motion** of a projectile:

$$\mathbf{v_y = v_{0y} - gt}$$

$$\mathbf{y = y_0 + v_{0y}t - \tfrac{1}{2}\, gt^2}$$

$$\mathbf{v_y^{\,2} = v_{0y}^{\,2} - 2g\,(y - y_0)}$$

6. The figure below shows the vertical and horizontal components of the soccer ball's projectile motion from the previous example:



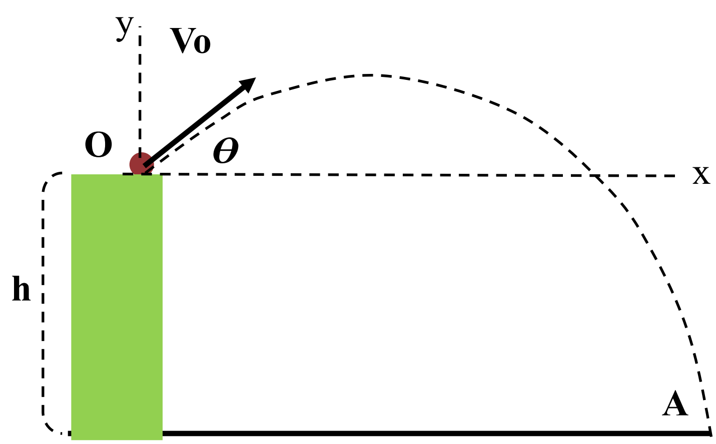
Vertical and horizontal components of projectile motion

Example 1

Given: A golf ball is hit off of an **h** = 170m cliff at point O at an angle of $\boldsymbol{\theta} = 47^0$ with an initial velocity $\mathbf{v_0}$ of 30 m/s.

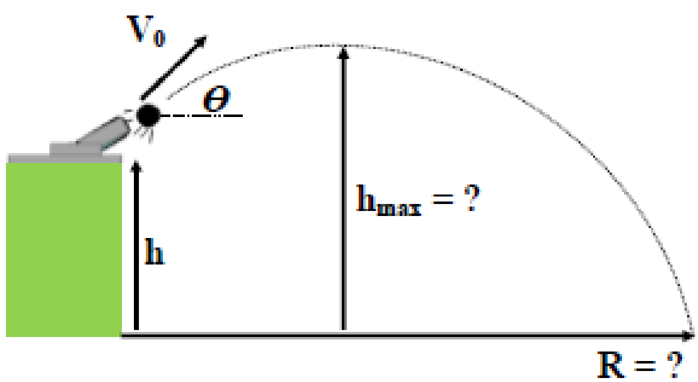
Find: a) The length of time the ball is in the air, and b) the velocity of the ball right before it hits the ground at point A.

Plan: Establish a fixed x,y coordinate system, making point O the origin. Apply kinematic equations in x and y-directions.



Lesson 4 Group Work

The cannon shoots a cannonball with a $\mathbf{v_0}$ of 34 m/s at an angle $\theta = 40°$ with respect to the x-axis. The cannon is located on a cliff h= 72 m higher than the region below. Determine the total time the cannonball is in the air, the total horizontal distance it travels, and the maximum height the cannonball reaches.



Plan:

Vertical Motion:

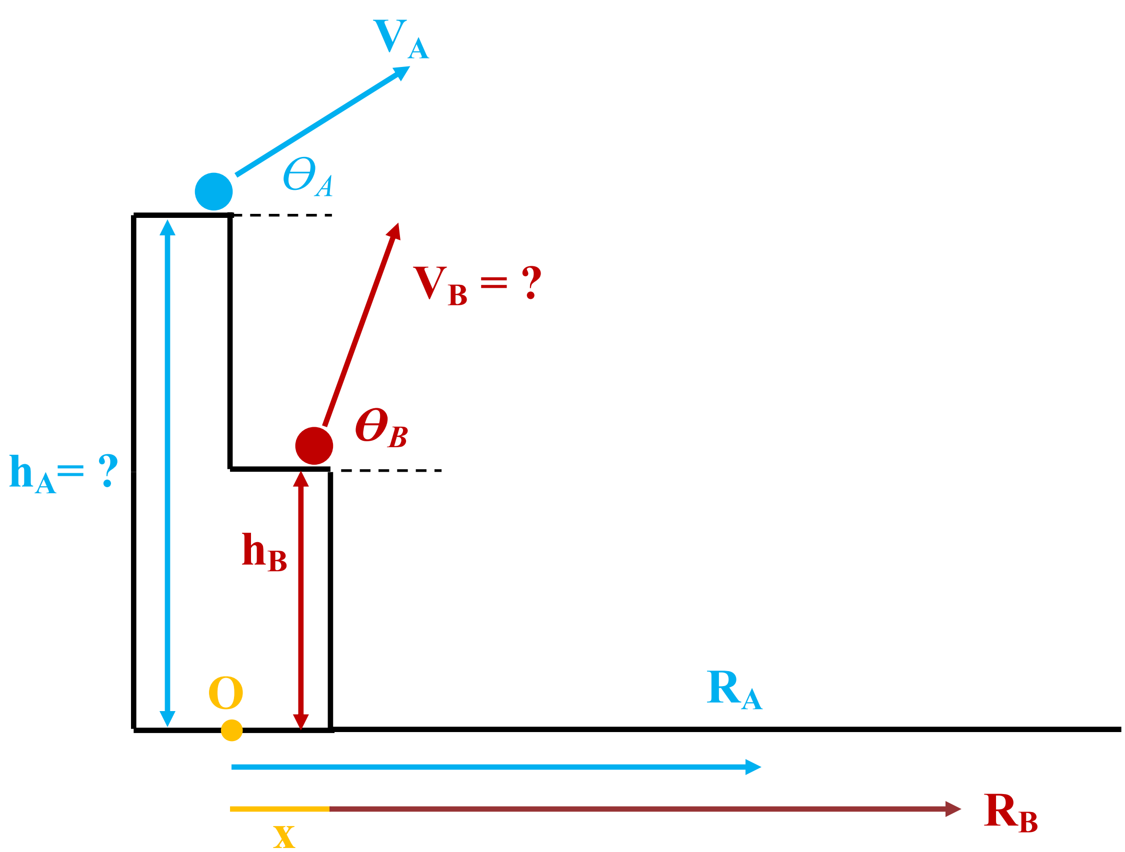
Horizontal Motion:

Homework Assignment 4

1. Balls A and B are launched from the structure shown at angles Θ_A and Θ_B in respect to the x-axis. Ball A has initial velocity V_A and travels a horizontal distance R_A. Ball B is launched from height h_B and travels horizontal distance R_B. Value x is given.

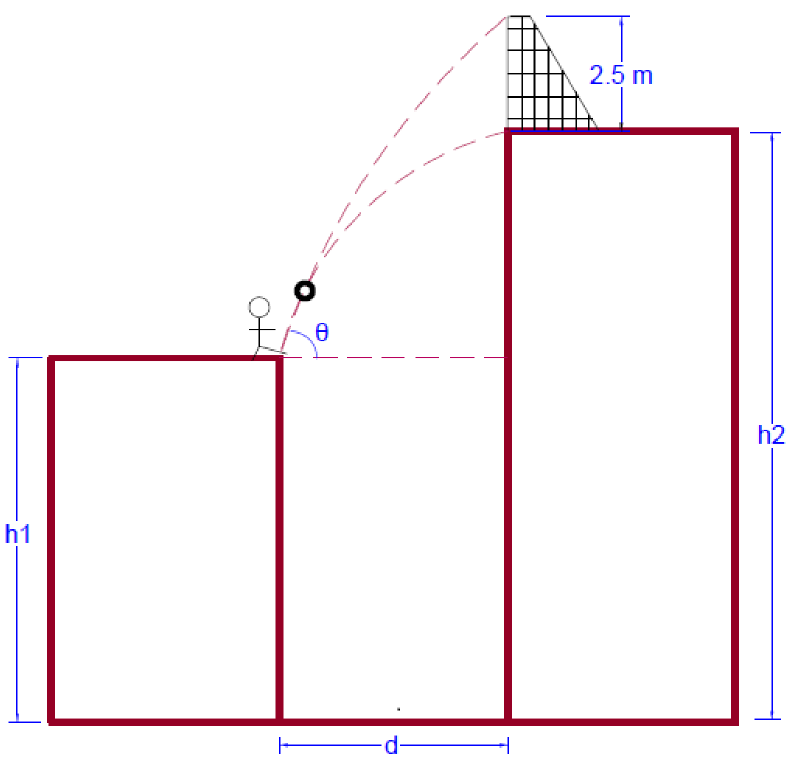
A) find the height from which Ball A is launched and the magnitude of the initial velocity of Ball B.

B) find the time when Balls A and B are at the same height and when they are at the same range from the origin. Calculate the height and range where this occurs.



Homework Assignment 4

2. Pele kicks a soccer ball from a building, aiming to shoot the ball into the net on the adjacent building. The goal is 2.5-m high. $\Theta = 45^\circ$, $h_1 = 20$ m, $h_2 = 30$ m, and the distance between the two buildings is d= 15 m. What are the minimum and maximum initial velocities required to make the goal?



Normal/Tangential Coordinates

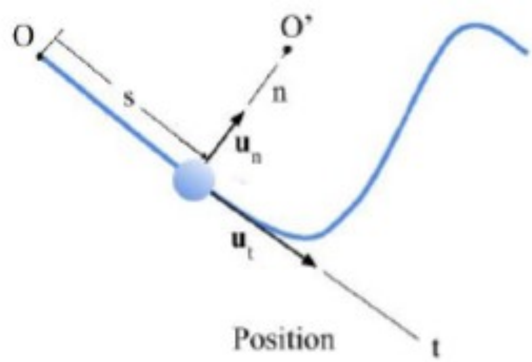
Lesson 5

1. The **normal and tangential coordinate system** is useful when a particle's path is curved and the motion is described along the path. When the radius of curvature and the particle's velocity or acceleration along the path are given, normal-tangential coordinates are an appropriate choice.

2. A sharply curved road is a good example of a curved path where the velocity of the vehicle is changing not only in magnitude, but also in direction. The components of acceleration will reflect changes in both magnitude and direction of velocity.

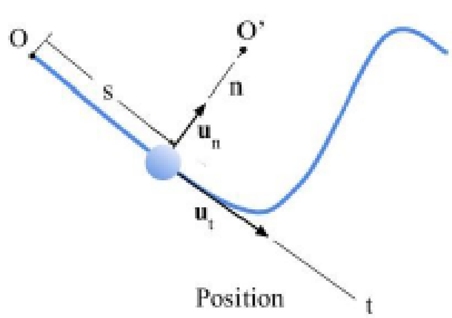
Curved Roadway Path

3. In the normal-tangential coordinate system, the origin is located on the particle. This means that the **origin moves with the particle.** The tangential axis is taken tangent to the path at any given instant. It is taken as positive in the **direction of the particle's motion.** The normal axis is perpendicular to the tangential axis with the positive direction taken **towards the center of curvature.**

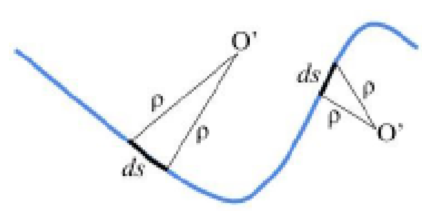


Position

4. The positive normal (n) and tangential (t) directions are defined by the unit vectors $\boldsymbol{u}_n$ and $\boldsymbol{u}_t$, respectively. The center of curvature, O', always lies on the concave side of the curve. The radius of curvature ρ is defined as the perpendicular distance from the curve to the center of curvature at that point. The position of the particle at any instant is defined by the distance, s, measured along the curve from a fixed reference point.

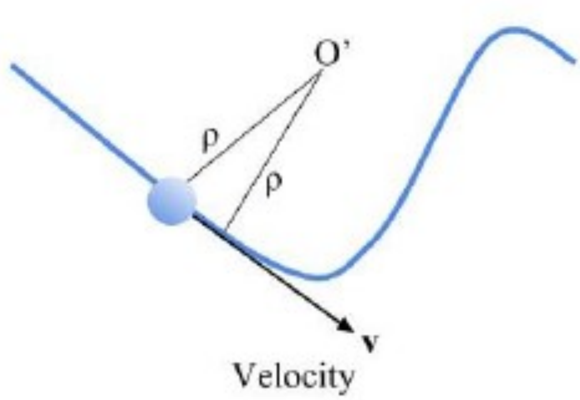


Position



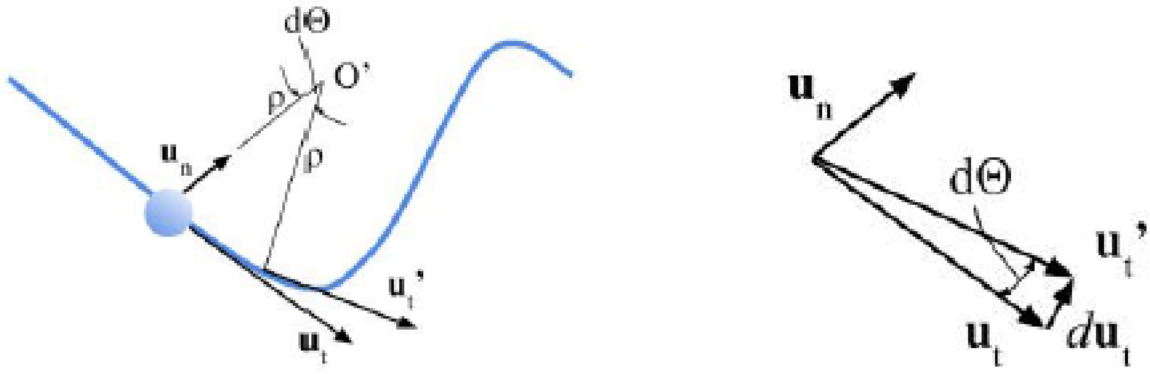
Radius of curvature

5. We know that the direction of the instantaneous velocity is **always** tangent to the path. Therefore, there will only be a tangential component of velocity. $\boldsymbol{v}$ = v $\boldsymbol{u}_t$, where v = ds/dt. Here v defines the magnitude of the velocity (speed) and $\boldsymbol{u}_t$ defines the direction of the velocity vector.



6. We differentiate the velocity vector to obtain the acceleration vector. After some manipulation, it can be shown that

$$\mathbf{a} = a_t \boldsymbol{u}_t + a_n \boldsymbol{u}_n = \dot{v}\, \boldsymbol{u}_t + (v^2/\rho)\, \boldsymbol{u}_n$$



Acceleration

7. So, there are two components to the acceleration vector:

$$\mathbf{a} = a_t \boldsymbol{u}_t + a_n \boldsymbol{u}_n$$

The **tangential component** is tangent to the curve and in the direction of increasing or decreasing velocity. It represents the change in speed.

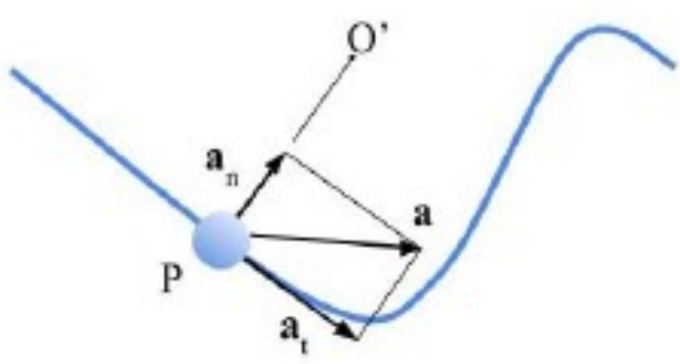
$$a_t = dv/dt$$

The **normal or centripetal component** is always directed toward the center of curvature of the curve. It represents the change in the velocity's direction.

$$a_n = v^2/r$$

The **magnitude** of the total acceleration vector is

$$a = [(a_t)^2 + (a_n)^2]^{0.5}$$



Acceleration

Special Cases of Motion

1) The particle moves along a straight line.

$\rho \longrightarrow \infty \quad \Rightarrow \quad a_n = v^2/\rho = 0 \quad \Rightarrow \quad a = a_t = dv/dt$

The tangential component is the only component of acceleration and represents the change in speed.

2) The particle moves along a curve at constant speed.

$a_t = dv/dt = 0 \quad \Rightarrow \quad a = a_n = v^2/\rho$

The normal component is the only component of acceleration and represents the time rate of change in the direction of the velocity.

3) The tangential component of acceleration is constant, $a_t = (a_t)_c$. In this case,

$s = s_o + v_o t + (1/2)(a_t)_c t^2$

$v = v_o + (a_t)_c t$

$v^2 = (v_o)^2 + 2 (a_t)_c (s - s_o)$

4) The particle moves along a path expressed as $y = f(x)$. The radius of curvature, ρ, at any point on the path can be calculated from

$$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{|d^2y/dx^2|}$$

Example 1

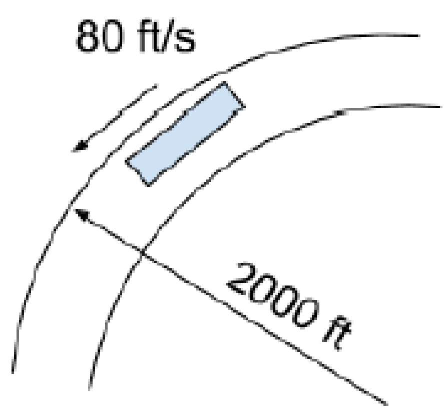
Given: A motorcyclist is traveling at 80 ft/s along a curved highway section with a radius of 2000 ft. The cyclist slows down at a constant rate in 10 seconds to a speed of 55 ft/s.

Find: The total acceleration after the motorcyclist slows down.

Plan: Calculate a_t using kinematic equations

Find a_n with the normal acceleration equation

Use the magnitudes of a_t and a_n to solve for the total acceleration

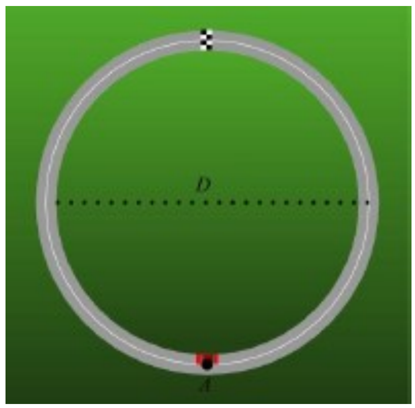


Example 2

At t= 0, an automobile begins to move with constant accelerating speed along a circular curve of radius 1500 ft. and acquires a speed of 20 mph after 60 seconds. Find the normal and tangential components of acceleration at t = 25 seconds.

Lesson 5 Group Work

A car is racing around a circular track. The car starts a lap with a speed of 100 m/s, accelerates uniformly, and finishes the lap with a speed of 150 m/s. The track has a diameter of 350 meters. Determine the magnitude of total acceleration halfway (Point A) thru the lap.



A

Plan:

Tangential component of acceleration:

Velocity at A:

Normal component of acceleration at A:

Magnitude of acceleration at A:

Homework Assignment # 5

1. A race car is driven around a circular track. A) If the speed of the car is 75 mph, and the normal component of acceleration is 25 ft/s^2, what is the diameter of the track? B) If the diameter of the track is 500 feet, and the normal component of acceleration is 0.5g, what is the speed of the car in mph?

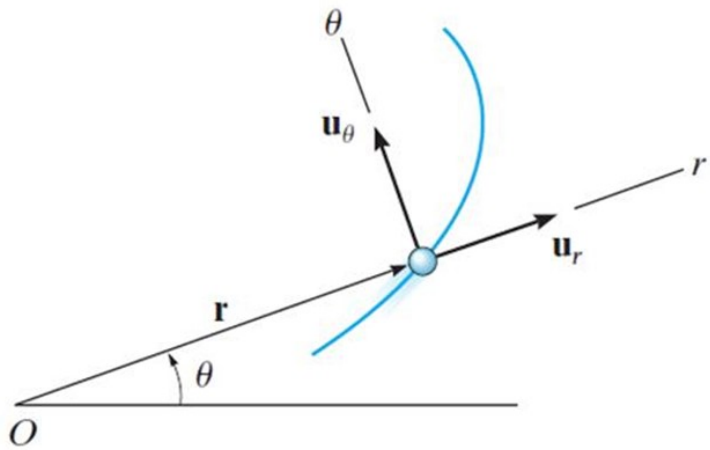
2. An indoor track is 100 meters in diameter. A runner increases his speed at a constant rate from 4 m/s to 8 m/s over a distance of 30 meters. Determine the total acceleration of the runner 3 seconds after he begins to increase his speed.

3. A boat travels in a circular path of radius 400 ft. Its speed varies as a function of time according to $v = 6(t^2 - t)$. Determine the magnitude of the boats total acceleration at $t = 2$ seconds and how far it has traveled in $t = 8$ seconds.

Cylindrical Coordinates

Lesson 6

1. The final coordinate system we will look at with respect to curvilinear kinematics is **polar** (two-dimensions) or **cylindrical** (three-dimensions) **coordinate systems**. Polar coordinates are useful when the path of the particle is known. This means that some of the information about the particle's distance from a fixed origin as well as its **rotation** about a fixed axis are given. A common example is circular motion when angular velocities and accelerations are given.



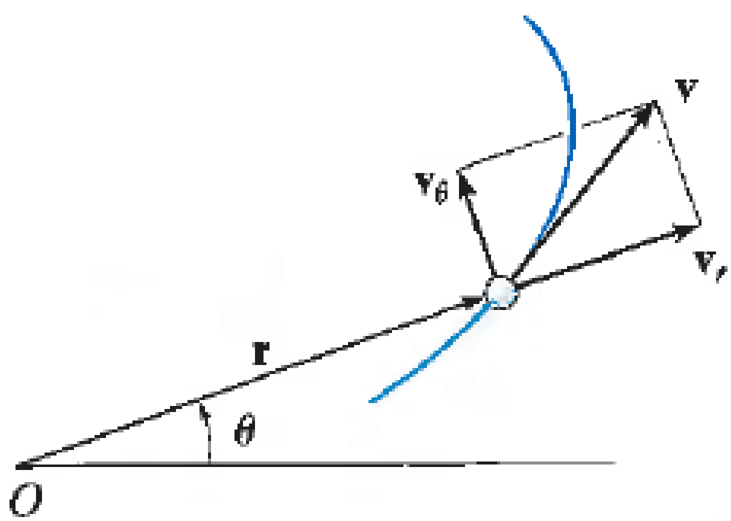
2. We can express the position of a particle moving in a curvilinear path in polar coordinates by a position vector $\mathbf{r} = r\,\mathbf{u_r}$, measured from a <u>fixed origin</u>. The transverse angular coordinate θ is measured counterclockwise from the horizontal.

3. We may obtain the instantaneous velocity by differentiating the position vector with respect to time. After some manipulation we can express the velocity vector in polar coordinates as:

$$\mathbf{v} = \dot{r}\,\mathbf{u_r} + r\,\dot{\theta}\,\mathbf{u_\theta}$$

The radial component of velocity is $\dot{r}$, and the transverse component of velocity is $r\,\dot{\theta}$. The magnitude of velocity (speed) is

$v = ((r\,\dot{\theta})^2 + (\dot{r})^2)^{1/2}$.

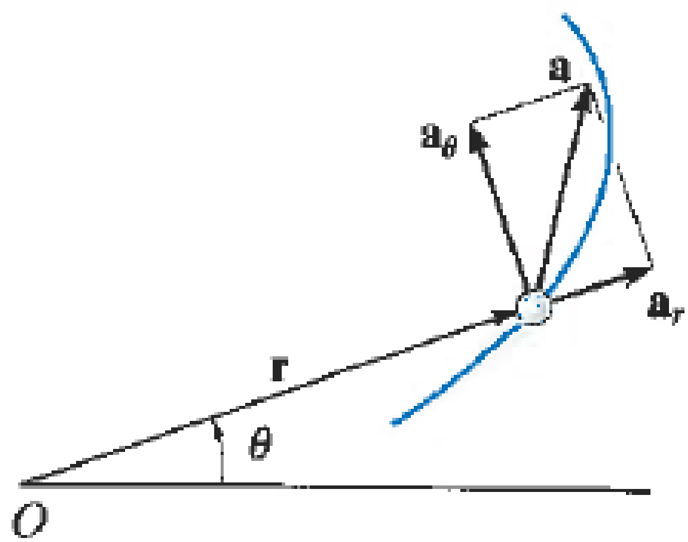


Velocity

4. We may obtain the instantaneous acceleration vector by differentiating the velocity vector with respect to time. After some manipulation we can express the acceleration vector in polar coordinates as:

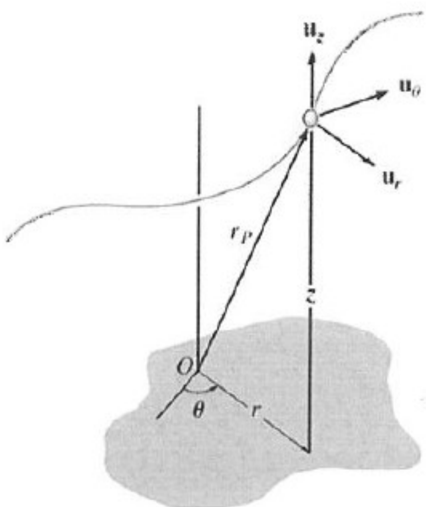
$$\mathbf{a} = (\ddot{r} - r\,\dot{\theta}^2)\,\mathbf{u_r} + (r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta})\,\mathbf{u_\theta}$$

The radial component of acceleration is $(\ddot{r} - r\,\dot{\theta}^2)$ and the transverse component of acceleration is $(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta})$. The magnitude of total acceleration is $a = [(\ddot{r} - r\,\dot{\theta}^2)^2 + (r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta})^2]^{1/2}$.



Acceleration

5. For three-dimensional problems, we add a z-term to the polar coordinate terms for position, velocity and acceleration. We refer to the three-dimensional coordinate system as cylindrical coordinates.



3D Position Coordinates

If particle P moves along a space curve, its position can be written as $\mathbf{r}_{\mathbf{P}} = r\,\mathbf{u_r} + z\,\mathbf{u_z}$.

Taking time derivatives and using the chain rule:

Velocity: $\mathbf{v_P} = \dot{r}\,\mathbf{u_r} + r\,\theta\,\mathbf{u_\theta} + \dot{z}\,\mathbf{u_z}$

Acceleration: $\mathbf{a_P} = (\ddot{r} - r\,\theta^2)\,\mathbf{u_r} + (r\,\theta + 2\,r\,\theta)\,\mathbf{u_\theta} + \ddot{z}\,\mathbf{u_z}$

The radial component of acceleration is $(r - r\,\theta^2)$ and the transverse component of acceleration is $(r\,\theta + 2\,r\,\theta)$. The mag-

Example 1

A particle is described by polar coordinates r = sin(t) m and $\Theta = e^t$ rad. Determine the radial and transverse components of the particle's velocity and acceleration when t = 4 sec.

Given: Position of the particle

Find: Velocity and acceleration of the particle

Plan: Compute the first and second derivatives of r and θ

Substitute into the equations for polar components

Group Work

The position of the particle is given in the polar coordinates as $r = 9t^3 + \cos(t)$ m and $\theta = 3t^2$ rad/s. Determine the radial & transverse components of acceleration of the particle at t = 3s if $\theta_{t=0} = 0.5$ radians. Assume there exists a second particle with $r = 6t^2 - \sin(t) + 14t$, find at what time t that the two particles have equivalent radial components.

Given: A) $r = 9t^3 + \cos(t)$ m, $\theta = 3t^2$ rad/s, $\theta = 0.5$ rad

B) $r = 6t^2 - \sin(t) + 14t$

Find: A) Acceleration of particle at t=3s

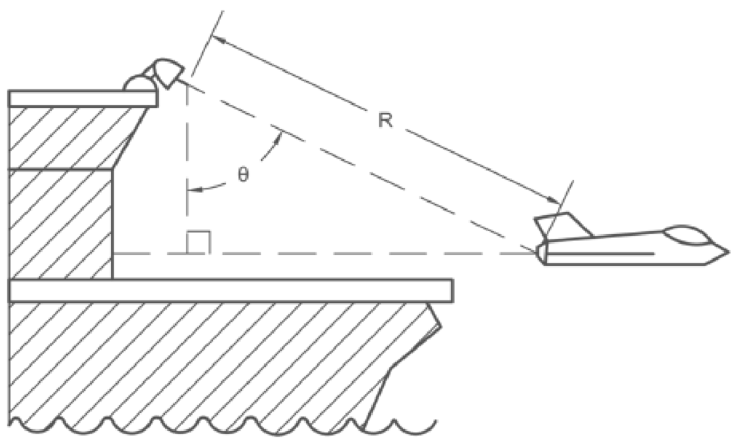
B) Time at which two particles have equal radial components

Plan:

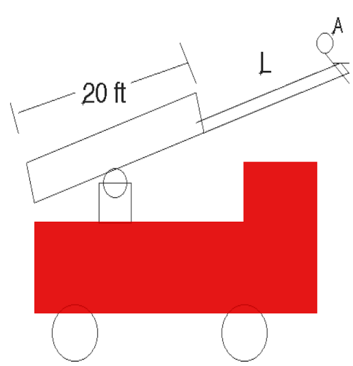
Solve for time t :

Homework Assignment # 6

1. A fighter jet is launched from an aircraft carrier. The aircraft is then tracked by radar on top of the command tower. When $\theta = 25^0$, $r = 54$m, $r = 60$ m/s^2, and $\dot{\theta} = 0.9$ rad/sec. Calculate the velocity and acceleration of the fighter jet in this position.



2. fire truck ladder extends at a constant rate of 9 in/sec and elevates at a constant rate of 4 degrees/sec. Determine the magnitude of velocity and acceleration of the fireman at A when $\theta = 55°$ and $L = 12$ ft.

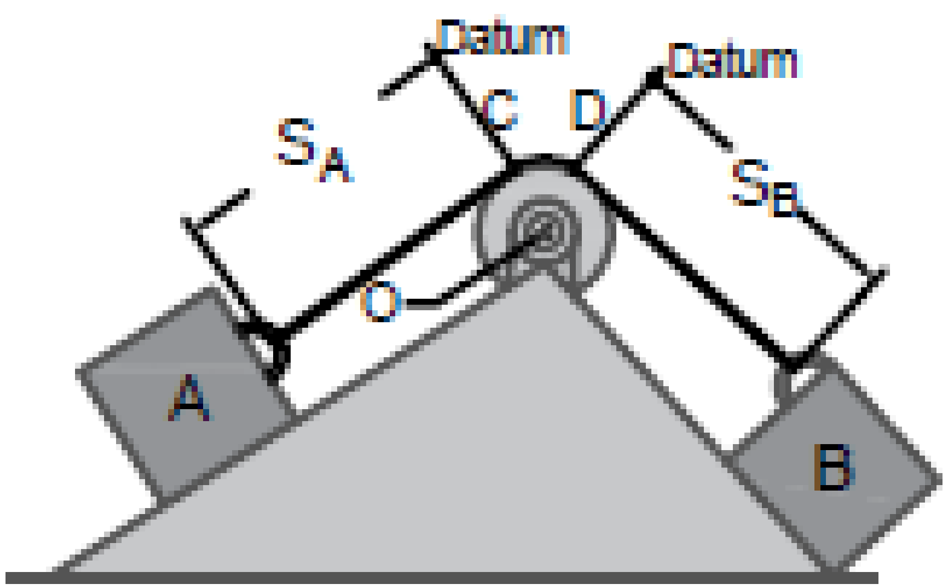


Dependent Motion

Lesson 7

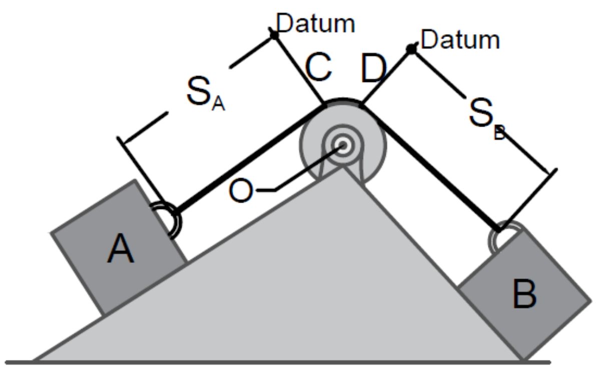
1. There are many problems in which the motion of two bodies is related to one another in some way. We refer to this class of problems as dependent motion problems.

2. One of the most common dependent motion problems is that of bodies connected by cords and pulleys. It is clear from observation that the motion of the two bodies is related. If block A moves down, block B must move up. By making the assumption that the cord connecting the two bodies is inextensible, we can develop a mathematical relationship between the positions of the two bodies.



Dependent motion of Blocks A and B

3. The first step is to select a <u>fixed datum</u> from which we can define position coordinates for each body, directed along each body's path of motion. We then write a cord equation in terms of the position coordinates. Because the cord is assumed inextensible the length of the cord is a constant.



Position coordinates

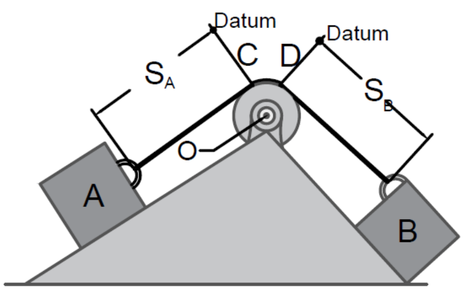
In this example, position coordinates s_A and s_B can be defined from fixed datum lines extending from the center of the pulley along each incline to blocks A and B.

If the cord has a fixed length, the position coordinates s_A and s_B are related mathematically by the equation

$$\mathbf{s_A + l_{CD} + s_B = l_T}$$

Here l_T is the total cord length and l_{CD} is the length of cord passing over the arc CD on the pulley.

4. We then differentiate the position equation to obtain the relationship between the two velocities. A second differentiation yields the relationship between the two accelerations. Algebraic signs are important, with the positive direction of position, velocity and acceleration taken as from the datum to the body.



Position coordinates

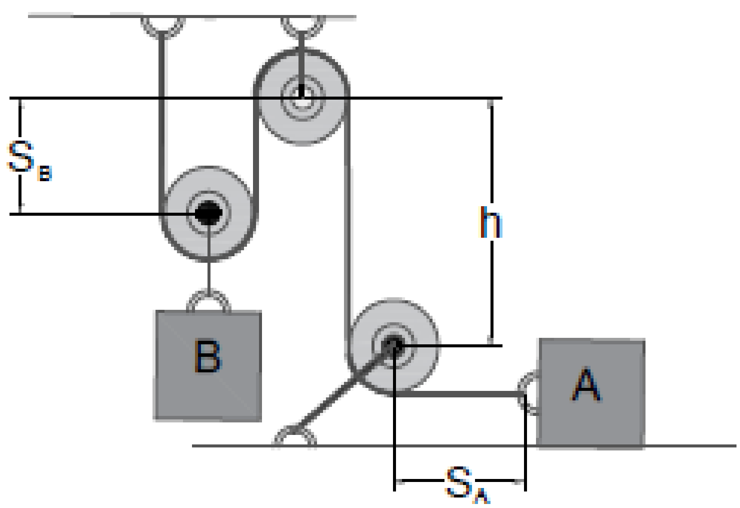
The velocities of blocks A and B can be related by differentiating the position equation. Note that l_{CD} and l_T remain constant, so $\mathbf{dl_{CD}/dt = dl_T/dt = 0}$

$$\mathbf{ds_A/dt + ds_B/dt = 0 \quad => \quad v_B = -v_A}$$

The negative sign indicates that as A moves down the incline (positive s_A direction), B moves up the incline (negative s_B direction).

Accelerations can be found by differentiating the velocity expression. In this case, $\mathbf{a_B = -a_A}$.

5. Let's illustrate this procedure by means of an example. In this case, the upper pulley is fixed and serves as a logical datum for body B, the lower pulley is fixed and serves as a logical datum for body A. The pulley above body B moves with body B.



Position coordinates—Pulley example

Position coordinates (s_A and s_B) are defined from fixed datum lines, measured along the direction of motion of each block.

Note that s_B is only defined to the center of the pulley above block B, since this block moves with the pulley. Also, h is a constant.

6. We write the equation for the total length of the cord. Knowing that we are going to differentiate this expression to obtain relationships between velocities and accelerations, we do <u>not</u> need to explicitly include any segment lengths of the cord that remain constant. Two differentiations of the position equations yield the desired relationships between the velocities and accelerations of the two bodies.

$$\text{Position: } 2s_B + h + s_A = l_T$$

$$\text{Velocity and Acceleration: } 2v_B = -v_A \quad \text{and} \quad 2a_B = -a_A$$

7. The step by step procedure for solving dependent motion problems with cords and pulleys is summarized here.

Dependent Motion Procedures:

1. Define position coordinates from fixed datum lines, along the path of each particle. Different datum lines can be used for each particle.

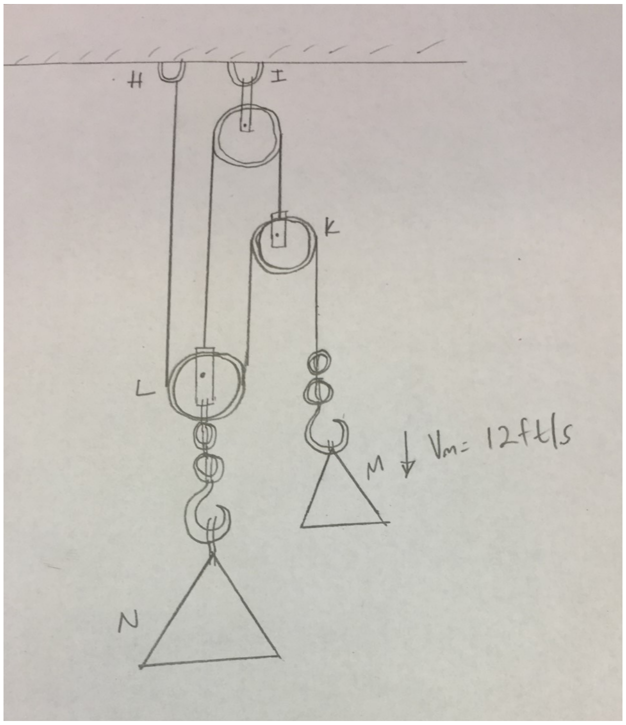
2. Relate the position coordinates to the cord length. Segments of cord that do not change in length during the motion may be left out.

3. If a system contains more than one cord, relate the position of a point on one cord to a point on another cord. Separate equations are written for each cord.

4. Differentiate the position coordinate equation(s) to relate velocities and accelerations.

These procedures can be used to relate the dependent motion of particles moving along rectilinear paths (only the magnitudes of velocity and acceleration change, not their line of direction).

Example 1



Given: In the figure on the left, the cord at M is pulled down with a speed of 12 ft/s.

Find: The speed of block N.

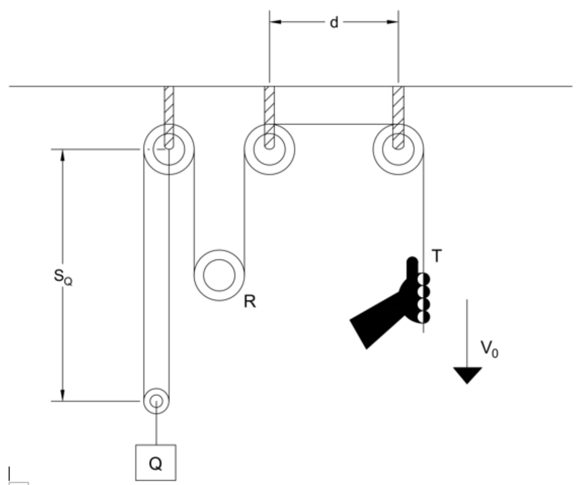
Plan: There are two cords involved in the motion in this example. There will be two position equations (one for each cord). Write these two equations, combine them, and then differentiate them.

Example 2

Given: In this pulley system, cord T is pulled down with a speed of 3 ft/s. Q moves up with a speed of 2 ft/s.

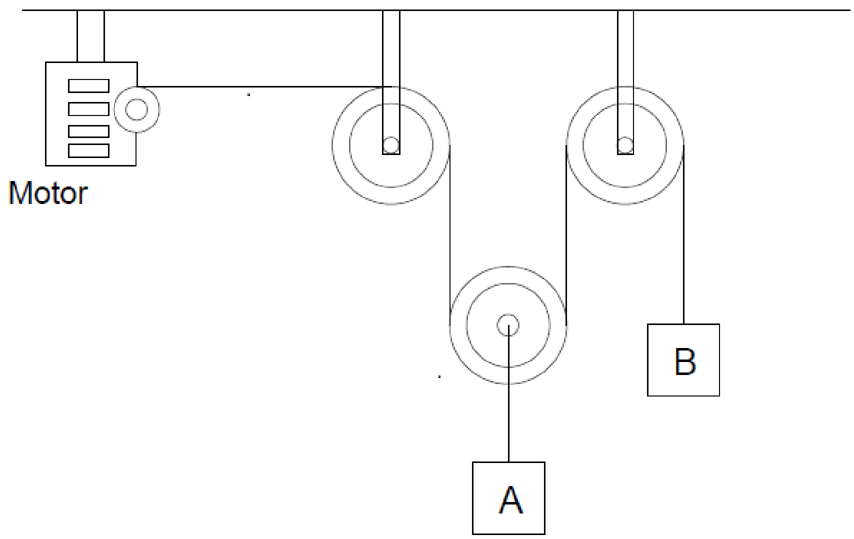
Find: The speed of pulley R.

Plan: Establish a datum. Write the cord equation. Differentiate the cord equation. Substitute the known speeds to solve for the speed of pulley R.



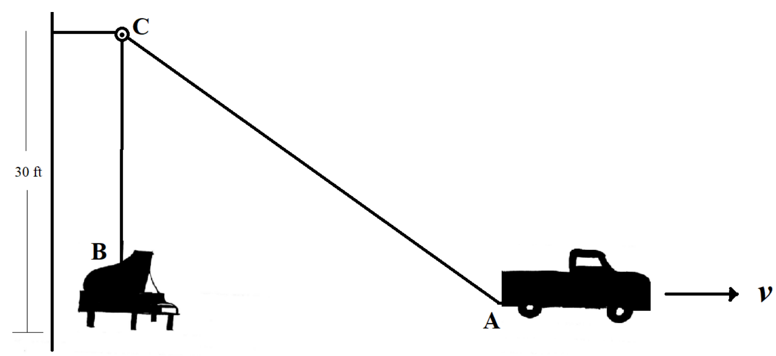
Lesson 7 Group Work

If the motor accelerates the rope from rest at a constant acceleration $a = 1.5$ m/s^2 for $t = 8$ seconds, the speed of block B is 12 m/s upward. Find the instantaneous speed of block A.

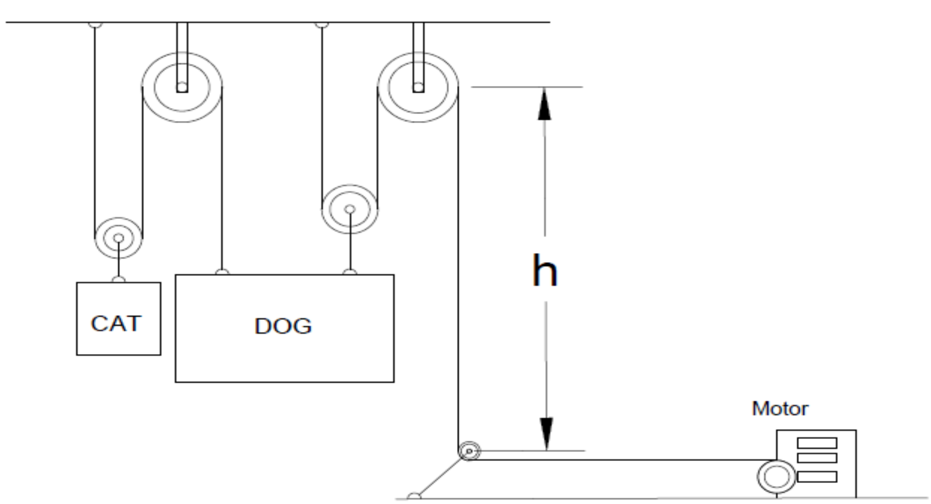


Homework Assignment # 7

1. A truck at point A is used to lift a grand piano at B using an 80 ft rope and a pulley at C. The truck moves with a constant velocity of 5 mph. Find the velocity of the piano when it has been lifted 15 feet.



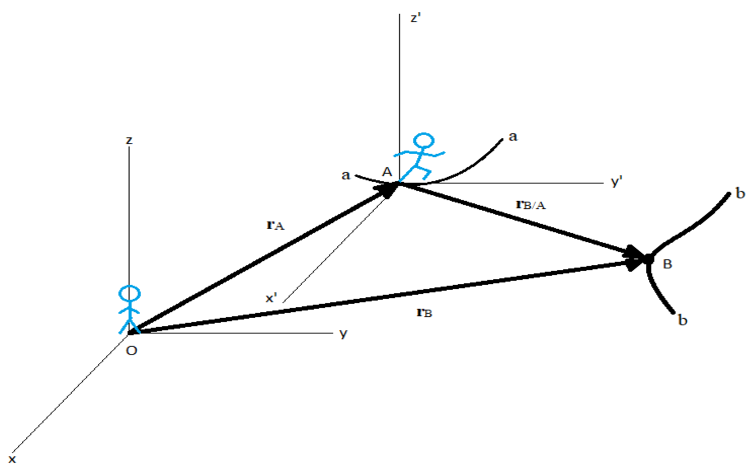
2. If the motor pulls in
the rope at a constant acceleration of $a_{motor} = 30$ ft/s^2 starting from rest, find the velocities of the Cat and the Dog at t = 7 seconds.



Relative Motion

Lesson 8

1. Up until now, we have considered the motion of a particle with respect to a fixed coordinate system. However, in reality, we never observe motion with respect to a fixed frame of reference. All of our coordinate axes are moving with the earth. Often that has no practical significance. However, it is sometimes necessary to investigate the motion of a particle relative to another moving particle.

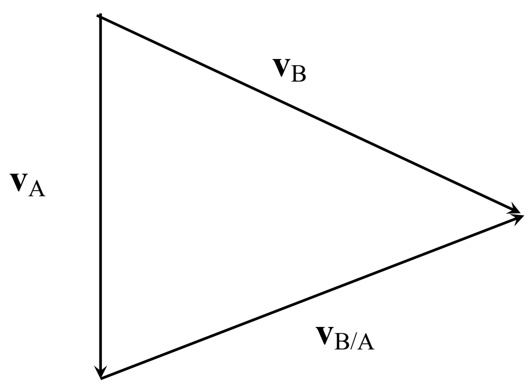


Relative position vector

2. This diagram shows two objects in motion, A and B. From a fixed origin (fixed observer), their motions are described by position vectors $\mathbf{r_A}$ and $\mathbf{r_B}$. We can define the relative position of B with respect to an observer located on A as $r_{B/A}$. From the resulting triangle of position vectors we can see that $\mathbf{r_B} = \mathbf{r_A} + \mathbf{r_{B/A}}$ (the absolute position of B is equal to the absolute position of A plus the relative position of B with respect to A.) If we know the absolute positions of the two particles, we can determine the relative position of B with respect to A by the equation

$$\mathbf{r_{B/A}} = \mathbf{r_B} - \mathbf{r_A}.$$

3. By differentiating the position equation, we get the relationship between absolute velocities and relative velocity as $\mathbf{v_B} = \mathbf{v_A} + \mathbf{v_{B/A}}$. Note the importance of the order in the relative velocity definition. $\underline{\mathbf{v_{B/A}}}$ <u>is the velocity of B with respect to A</u>, and is different than $\mathbf{v_{A/B}}$.

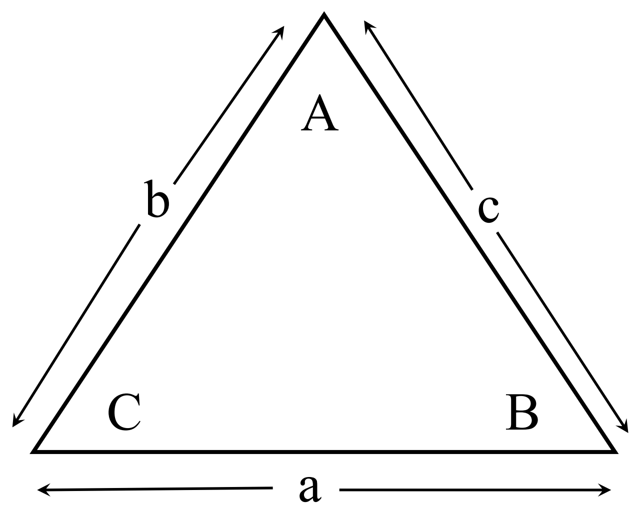


Relative velocity vector

4. By differentiating the velocity equation, we obtain a similar equation of relative acceleration $\mathbf{a_B} = \mathbf{a_A} + \mathbf{a_{B/A}}$.

5. Because these three vectors always form a triangle, we can solve relative motion problems two ways. First, we can express each vector in **Cartesian vector form**, and perform the necessary calculations. Secondly, we can make use of the law of sines and the law of cosines to perform a **graphical solution** to the problem.

6. The Law of Sines and the Law of Cosines my be used to determine length of sides (magnitude) and angles (direction).



Law of Sines and Cosines

Law of Cosines: $a^2 = b^2 + c^2 - 2bc \cos A$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

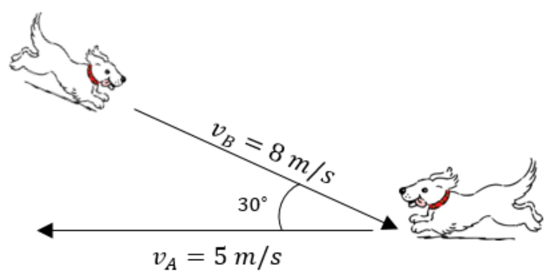
Example 1

Given: The absolute velocities of a two puppies are 8 m/s and 5 m/s. The direction of motion is shown in the diagram below.

Find: $\mathbf{v}_{B/A}$

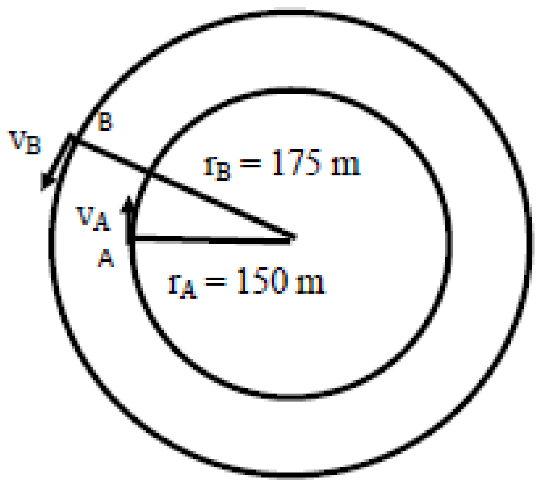
Plan: Vector Method: Write $\mathbf{v}_A$ and $\mathbf{v}_B$ in CVN, then deter mine $\mathbf{v}_B$-$\mathbf{v}_A$.

Graphical Method: Draw vectors $\mathbf{v}_A$ and $\mathbf{v}_B$ from a common point. Apply the laws of sines and cosines to determine $\mathbf{v}_{B/A}$.



Lesson 8 Group Work

Point particles A and B are moving along circular paths. At the instant shown, A has a speed of 100 m/s and is increasing its speed at the rate of 10 m/s^2 , whereas B has a speed of 120 m/s and is decreasing its speed at 15 m/s^2 . Determine the relative velocity and relative acceleration of point particle A with respect to point particle B at this instant.



Plan:

Express $\mathbf{v_A}$ and $\mathbf{v_B}$ in CVN

Find $\mathbf{v}_{\mathbf{A/B}}$ in CVN

Find magnitude and direction of $\mathbf{v}_{\mathbf{A/B}}$

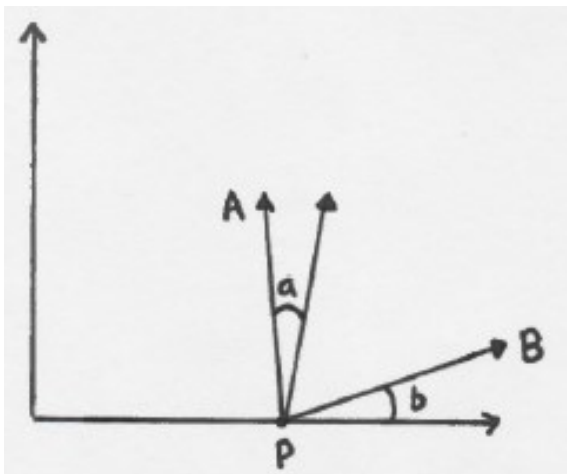
Express $\mathbf{a}_A$ and $\mathbf{a}_B$ in CVN

Find $\mathbf{a}_{A/B}$ in CVN

Find magnitude and direction of $\mathbf{a}_{A/B}$

Homework Assignment #8

1. Two tractors, A and B, leave a cornfield at the same instant and location. Tractor A travels at $v_A = 3$ m/s and tractor B travels at $v_B = 5$ m/s. Angle a = 10^0 and angle b = 25^0. Determine a) the velocity and direction of A with respect to B, and b) after 15 seconds, how far apart will the tractors be?



2. A ship is sailing south at a speed of 10 mph and is encountering a westerly current of 5 feet per second. What is the actual velocity of the ship, and how far will it be carried off course in 8 hours?

Equation of Motion- Rectangular Coordinates

Lesson 9

1. We now have the ability to describe the motion of a particle in rectangular, normal/tangential, and polar coordinates. We can therefore apply what we have studied of particle kinematics to the more interesting field of **particle kinetics**. Using particle kinetics, we can relate the forces acting on the particle to the particle's resulting motion. In this lesson, we will focus on relating forces and motion using **rectangular coordinates**.

2. Recall back to our study of Statics, in which the principles of equilibrium were based on Newton's First and Third Laws, which state that a body at rest has all externally-applied forces in balance.

Now in Dynamics, we use **Newton's Second Law**:

> If a particle has **unbalanced** externally-applied forces, then the particle experiences an **acceleration** in the direction of the **resultant force**. The magnitude of the acceleration is also proportional for the resultant force.

Newton's Second Law is referred to as the **equation of motion** and is mathematically defined as:

$$\mathbf{F} = m\mathbf{a}$$

Newton's Laws of Motion

First Law: A particle originally at rest, or moving in a straight line at constant velocity, will remain in this state if the resultant force of all externally-applied forces acting on the particle equals zero.

Second Law: If this resultant force is not zero, then particle experiences an acceleration in the same direction as the resultant force. This acceleration has a magnitude proportional to the resultant force.

Third Law: Mutual forces of action and reaction between two particles are equal, opposite, and collinear.

3. In dynamics, a common source of confusion is the difference between weight and mass. **Weight** is a force, and is equivalent to a body's mass times its acceleration due to gravity, or **W = mg**. In SI units, g equals 9.81 m/sec^2, and in Imperial units, g equals 32.2 ft/sec^2. **Mass** is an inertial property measuring a body's resistance to change in motion. It is the ratio between a body's resultant force and its acceleration due to gravity, or **m = F/a**.

$$\mathbf{W = mg}$$

$$\mathbf{m = F/a = W/g}$$

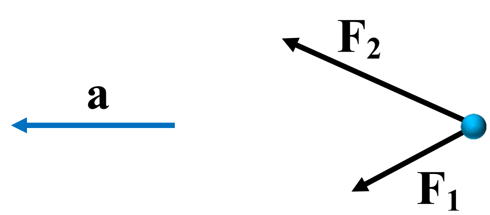
4. In dynamics, we draw two separate diagrams to apply the equations of motion — a **free-body diagram** and a **kinetic diagram.** The free-body diagram shows all of the external forces acting on the particle. The kinetic diagram shows the inertial force m**a**, which acts in the same direction as the resultant force.

If more than one force acts on the particle, the equation of motion can be written:

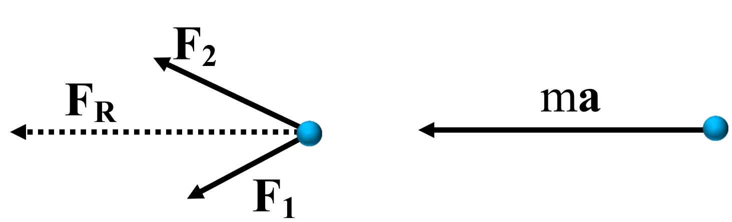
$$\boldsymbol{F} = \boldsymbol{F}_{\mathrm{R}} = m\mathbf{a}$$

where $\boldsymbol{F}_{\mathrm{R}}$ is the resultant force, which is a vector summation of all the forces.

To illustrate the equation, consider a particle with two external forces and acceleration in the direction shown:



Draw the Free-Body Diagram with all external force vectors acting on the particle. To draw the Kinetic Diagram, find the direction of the resultant force $\mathbf{F}_{\mathbf{R}}$, then draw the inertial force m**a** in the same direction.



Free-Body Diagram　　　　Kinetic Diagram

5. Procedure for Applying the Equation of Motion

A) Select a **convenient coordinate system**. Rectangular, normal/tangential, or cylindrical coordinates may be used.

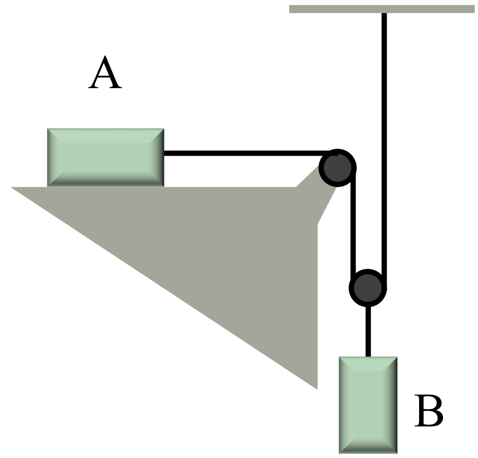
B) Draw a **free-body diagram** showing all external forces applied to the particle. Resolve the forces into their appropriate components.

C) Draw the **kinetic diagram**. Find the resultant force from the FBD and show the particle's inertial force, m**a** in its direction. Resolve the inertial force vector into its appropriate components.

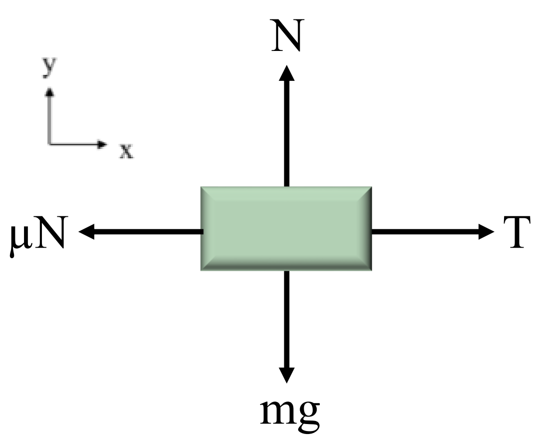
D) Apply the **equations of motion** in their scalar component form and solve these equations for the unknowns.

E) It may be necessary to use **kinematic equations** to determine other desired parameters of motion.

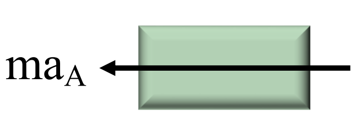
6. The two blocks shown at the right each have mass m. The coefficient of kinetic friction is μ. Block A is moving to the right.



7. Shown below are the free-body diagrams and kinetic diagrams for Blocks A and B.

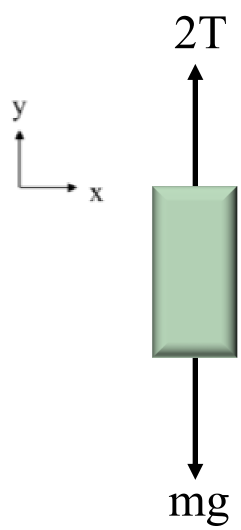


FBD

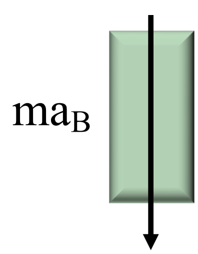


Kinetic

Block A



FBD



Kinetic

Block B

8. In rectangular coordinates, the equation of motion $\sum \mathbf{F} = m\mathbf{a}$ can be broken down into x y and z scalar components:

$$\sum F_x = ma_x \qquad \sum F_y = ma_y \qquad \sum F_z = ma_z$$

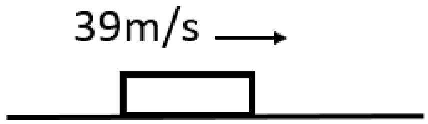
These can also be written in cartesian vector notation:

$$\sum F_x \boldsymbol{i} + \sum F_y \boldsymbol{j} + \sum F_z \boldsymbol{k} = m\,(a_x \boldsymbol{i} + a_y \boldsymbol{j} + a_z \boldsymbol{k})$$

9. Note that each equation of motion only yields acceleration. If a problem requires finding position or velocity, use kinematic equations to calculate them from the acceleration.

Example 1

A hockey puck on a frozen pond is given an initial speed of 39 m/s. If the puck always remain on the ice and slides 216 m before coming to rest, determine the coefficient of the kinetic friction (μ_k) between the puck and ice.



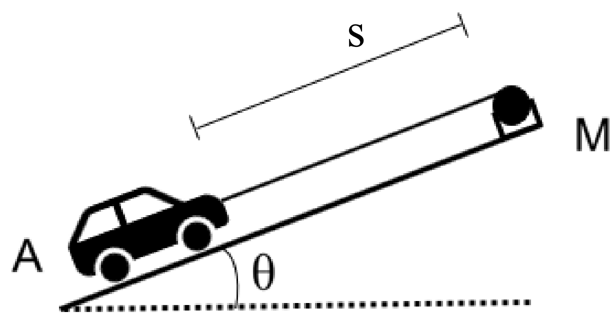
Given: $V_0 = 39$ m/s and s = 216 m

Find: Determine the coefficient of kinetic friction (μ_k) between the puck and ice.

Plan: Draw the free-body diagram and kinetic diagram of the puck. Use Newton's Law of Motion to solve for a_x, and use the kinematic equation $v^2 = v_0^2 + 2a_c(s-s_0)$ to solve for μ_k

Lesson 9 Group Work

A motor winds in a cable with a constant acceleration such that a 1500 kg car is moved a distance of s = 8 m in 5 seconds, starting from rest on an incline of $\theta = 40°$. $\mu_k = 0.4$. Find the tension developed in the cable.



Plan:

Free Body and Kinetic Diagrams:

Acceleration of car pulled up ramp:

Equation of Motion, determine normal force

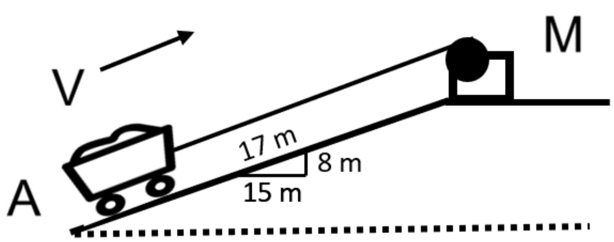
Equation of Motion, determine the tension:

Homework Assignment # 9

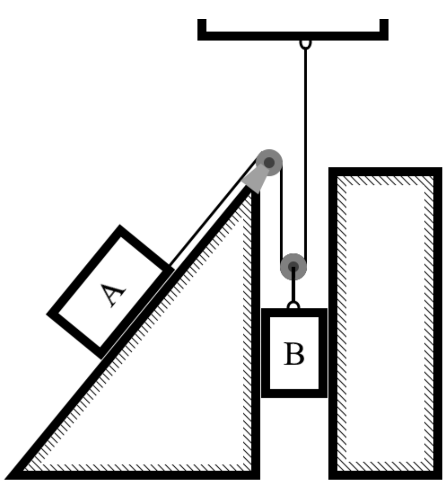
1. Mine cart A of mass $\mathbf{m}$ = 400 kg is pulled up by motor M on the incline shown. The tension in the cable is defined by:

$$\mathbf{F} = 3200t^2 \text{ N.}$$

The cart has initial velocity $\mathbf{v_0}$ = 2 m/s. Find the velocity of the cart at $\mathbf{t}$ = 2 seconds.



2. Block A, which is connected to block B by a cord, is sliding down a hill at an angle Θ = 53º. The hill has a coefficient of friction of μ = .28, while the shaft that block B is sliding up is frictionless. Block A has a mass of 12 kg and block B has a mass of 6 kg. Find V_B after block A slides 5m down the hill. System starts from rest.



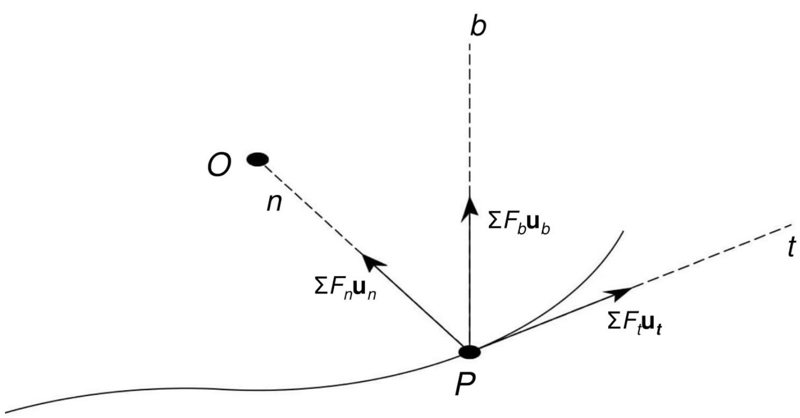
Equation of Motion-N/T Coordinates

Lesson 10

1. In Lesson 5, we discussed when and how to use the **normal/ tangential** coordinate system for curvilinear motion problems. In this lesson, we build on that concept by utilizing the normal/tangential coordinate system in conjunction with the **equations of motion** for a particle.

2. In kinematics, the **tangential direction** lies tangent to the path of motion, with the positive direction being in the direction of motion. The **normal direction** lies normal to the tangential direction, with the positive direction <u>always</u> pointing towards the center of curvature. For three-dimensional situations, the third direction is normal to the plane of the other two axes, also known as the **binormal direction**.

The normal direction (n) <u>always</u> points toward the path's center of curvature. The tangential direction (t) is tangent to the path, usually set as positive in the direction of motion of the particle.



Normal and tangential coordinate axes

3. We start with the same vector equation of motion used with rectangular coordinates, $\sum \mathbf{F} = m\mathbf{a}$. In the normal/tangential coordinate system, this provides us with two scalar equations of motion: $\sum F_t = ma_t$ and $\sum F_n = ma_n$. For three-dimensional applications, there is no motion in the binormal direction, so $\sum F_b = 0$.

4. From kinematics, the **tangential acceleration** represents the **change in <u>magnitude</u>** of the velocity vector, or the change in speed, $a_t = dv/dt$. If the particle is speeding up, a_t will be in the direction of the positive t axis, and if the particle is slowing down, a_t will be in the direction of the negative t axis.

The **normal acceleration** represents the **change in the <u>direction</u>** of the velocity vector, and is always pointing towards the center of curvature, in the positive n direction. The magnitude of normal acceleration is v^2/ρ, where ρ is the radius of curvature. Since the normal acceleration always points towards the center of curvature, the normal force that causes the normal acceleration will also always be pointing towards the center of curvature.

For curvilinear paths defined by mathematical functions, the radius of curvature at any point may be found by the formula:

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$

Procedure for Solving Problems with n-t Coordinates

1. Use n-t coordinates when a particle is moving along a known, curved path.

2. Establish the n-t coordinate system on the particle.

3. Draw free-body and kinetic diagrams of the particle. The normal acceleration (a_n) always acts "inward" (the positive n-direction). The tangential acceleration (a_t) may act in either the positive or negative t direction.

4. Apply the equations of motion in scalar form and solve.

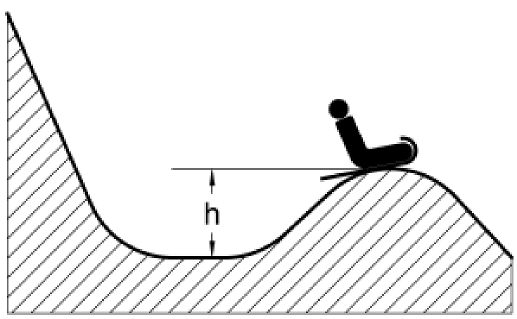
5. It may be necessary to employ the kinematic relations:

$$a_t = dv/dt \quad a_n = v^2/\rho$$

Example 1

Given: A child with a mass of 45kg is sledding down a hill and encounters a bump. The height h of the bump is 10 ft. (ρ=h)

Find: The maximum speed the sled can go over the bump without losing contact with the ground.



Plan:

1) Draw the sled's free-body and kinetic diagrams.

2) Solve for the velocity when the sled is just staying on the ground (N=0)

3) Apply the equation of motion in the n and t directions.

Example 2

A ferris wheel with a radius of 12 m has eight cars that are pin-connected to the wheel. It is turning at a speed of 4 m/s. A 50 kg-child is in one of the cars. Neglecting the weight of the cars, calculate the normal and tangential reactions of the child's car at the point when the car is 60^0 above the horizontal.

Given: $r = 12m$, $\theta = 60^0$, $m = 50$ kg, $v = 4$ m/s, $a_t = 1.3$ m/s^2

Find: The reaction forces F_n and F_t of the ferris wheel seat on the child

Plan:
1) Using n/t coordinates draw FBD and kinetic diagrams of the seat
2) Apply the equations of motion in the n and t directions

Lesson 10 Group Work

A motorcyclist is traveling at a constant velocity V, a distance d away from a uniform circular curve of radius r. The curve has a bank angle of θ.

Given: V=12 m/s, r=50 m, θ=5°, d=30 m

Find: a) The maximum speed at which the motorcycle can go through the curve.

b) The required acceleration of the motorcycle to achieve the speed found in part (a) by the start of the curve.

Plan

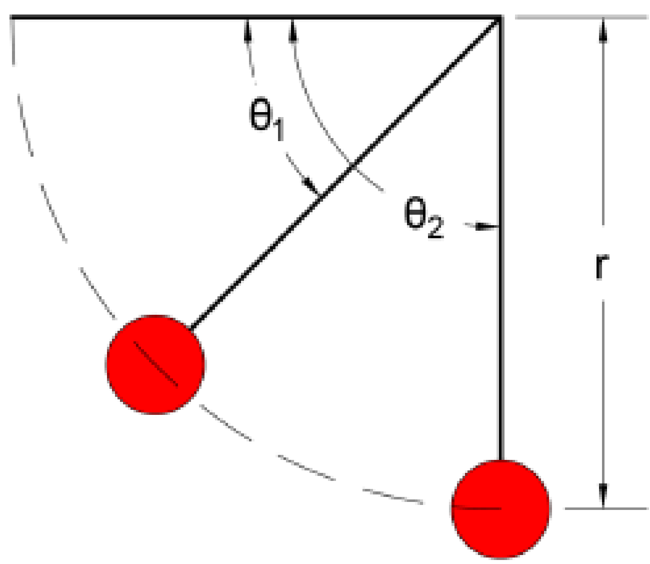
Draw a FBD and a kinetic diagram of the motorcycle of the curve

Use $\sum F_n = ma_n$ to solve for V_{max}

Use constant acceleration equations to solve for the required acceleration

Homework Assignment # 10

1. A wrecking ball attached to a cable swings freely in a vacuum (neglect air resistance). The cable length r is 15m, and the ball has a mass of 450 kg. When the wrecking ball is first released, the angle between the cable and the roof is $\theta_1 = 35^0$. Find the velocity of the ball and tension in the cable when θ_2 reaches 90^0.

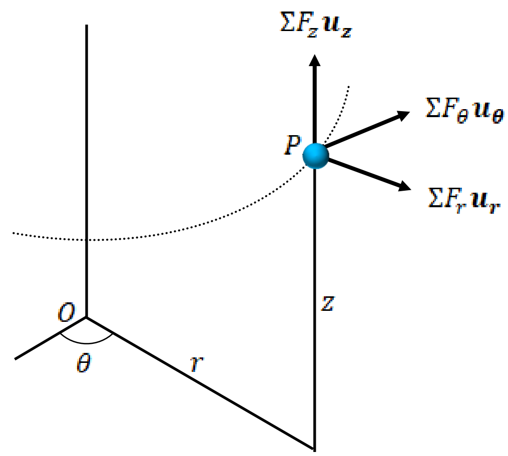


2. A race car drives around a circular track that has a radius 250 m. At a certain point, it has a velocity of 7 m/s, and the velocity is increasing ata rate of 0.12t m/s^2. Determine the aped of the car and its acceleration after it has traveled an additional one third of the way around the track.

Equation of Motion– Polar Coordinates

Lesson 11

1. The final coordinate system we will be using to apply the equations of motion is that of **polar** or **cylindrical coordinates.** Polar coordinates are preferable when the motion of a radial line from a fixed origin is defined angularly, such as for particle P in the figure shown:



Cylindrical coordinate axes

The general procedure for solving problems will be similar to that of both rectangular and normal/tangential coordinates. In Lesson 6 we learned the following kinematic expressions for acceleration in polar coordinates:

$$a_r = \ddot{r} - r\dot{\theta}^2 \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \qquad a_z = \ddot{z}$$

We can extract the equations of motion from Newton's Second Law **ΣF** = m**a**, resulting in the components:

$$\Sigma F_r = ma_r \qquad \Sigma F_\theta = ma_\theta \qquad \Sigma F_z = ma_z$$

Now we can combine these two to create the scalar equations of motion. In **cylindrical coordinates** (using *r*, *θ,* and *z* coordinates in three dimensions), they are expressed as:

$$\Sigma F_r = ma_r = m(\ddot{r} - r\dot{\theta}^2)$$

$$\Sigma F_\theta = ma_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

$$\Sigma F_z = ma_z = m\ddot{z}$$

If the particle is constrained to move only in the plane, where the problem is two-dimensional using **polar coordinates**, then only the radial and transverse coordinates are needed in the equations of motion:

$$\Sigma F_r = ma_r = m(\ddot{r} - r\dot{\theta}^2)$$

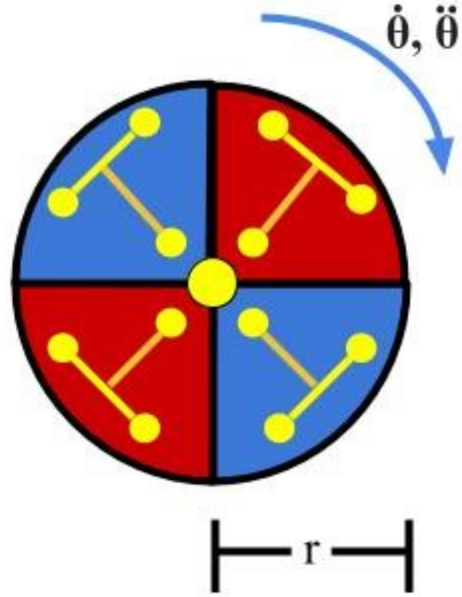
$$\Sigma F_\theta = ma_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

2. Remember, polar and cylindrical coordinates are based on a **fixed origin**. This is often confused with normal/tangential coordinates, where the origin moves with the particle.

Most fundamental problems using polar/cylindrical coordinates require you to determine the resultant force components that cause a particle to move with a known acceleration.

Example 1

A 50 kg child is riding a merry-go-round. It has a radius of 3.5 meters. It is rotating with an angular velocity of 2.5 rad/s and has an angular acceleration 4 rad/s². Determine the angular and radial force acting on the child. Neglect the mass of the merry-go-round.



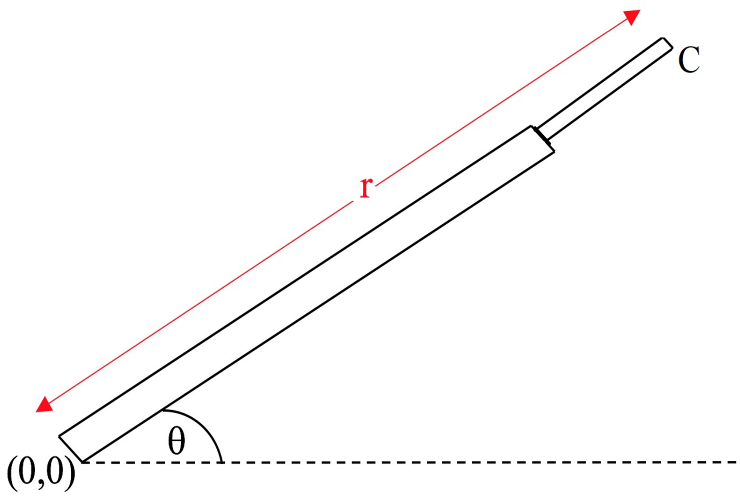
Given:

Find:

Plan:

Example 2

The polar coordinates of C, the end of the crane, are known functions of time.



Given: $r = 6 + 0.9t^2$ m , $\Theta = 0.06t^2$ rad.

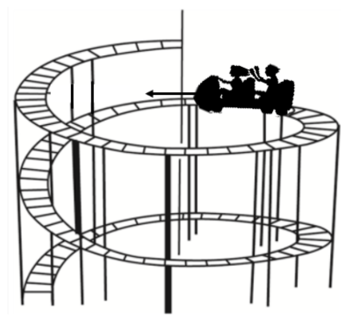
Find: Position, velocity and acceleration at t = 8 sec.

Plan:

Lesson 11 Group Work

Given: A 225-kg roller coaster car is traveling along the spiral track such that its position measured from the top of the track has components r = (8-t) m, Θ = (0.3t + 0.9) rad, and z = (16-t) m, where t is in seconds.

Find: The magnitudes of the components of force which the track exerts on the car in the **r**, **Θ**, and **z** directions at the instant **t = 2s**. Neglect the size of the car.



Plan:

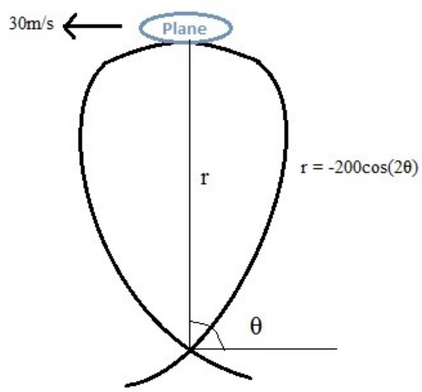
FBD of roller coaster car:

Kinematics:

Equations of Motion:

Homework Assignment # 11

1. A plane is flying with a constant velocity of 30m/s in a vertical loop described by $r = -200\cos(2\theta)$. Suppose the pilot experiences a normal force of 400N at the top of the loop. Determine the mass of the pilot.

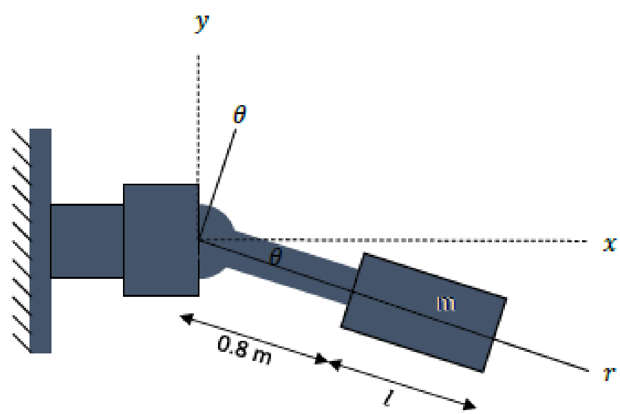


2. A rotating piston is both elevating and extending. At a specific instant,

$$\theta = 60°,\ \dot{\theta} = 80\ deg/s,\ \ddot{\theta} = 130\ deg/s^2,\ .$$

$$l = 0.8\ m,\ \dot{l} = 0.8\ m/s,\ \ddot{l} = -0.2\ m/s^2$$

Calculate the radial and transverse forces F_r and F_θ that the compression portion of the arm exerts on end of the arm, which has a mass of 2.2 kg. Compare with the case of static equilibrium in the same position.

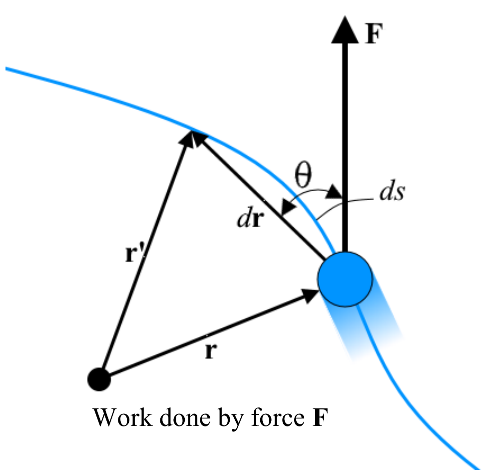


Principle of Work and Energy

Lesson 12

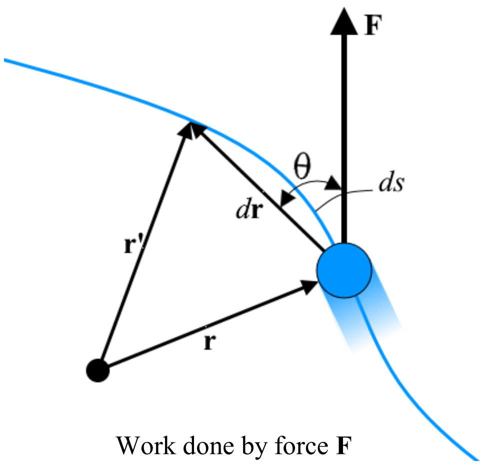
1. Let's look at particle kinetics from a different perspective. Although the **principle of work and energy** may seem like a new topic, it is simply the equation of motion expressed in a different form.

2. Before we apply the principle of work and energy, let's define the work that is done by a force. Work is a force that is applied over a distance. More specifically, work is achieved by force and displacement components acting in the same direction. The incremental work done by the force shown in the figure is the product of the component of the force acting along the path, or **dU = Fcosθ ds**. Remember, the dot product can be utilized to find the projection of one vector along the line of action of a second vector. In this case, the work done by the force F may be defined as the dot product between the force vector and the displacement vector. Integrating that dot product will give us the total work done by the force F as the particle displaces its initial position to is final position. Since we took the dot product of two vectors, work is a scalar quantity.



$$U_{1\text{-}2} = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$$

3. The general expression for work allows for the variation of both magnitude and direction of the force. For the special case of a constant force, the work expression simplifies to force multiplied by distance. If the force and displacement are in the same directions, positive work is done. If the force is opposed to the displacement, negative work is done.



Work done by force **F**

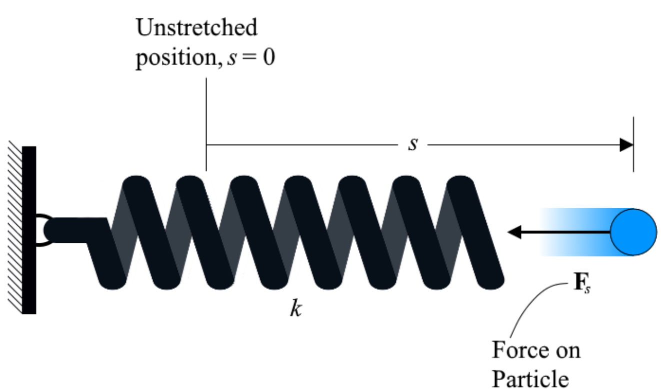
When F and θ are constant:

$$\mathbf{U_{1\text{-}2} = F_c \cos\theta (s_2 - s_1)}$$

4. Let's look at the special case of the work done by a weight force. The general work equation simplifies to –WΔy, where W is the constant weight force, and y is the vertical displacement (for this scenario, assume positive y is upwards). So, if a particle is displaced upwards, negative work will be done by the weight force, because the force is acting downwards.

$$U_{1\text{-}2} = \int_{y_1}^{y_2} -W\,dy = -W\,(y_2 - y_1) = -W\,\Delta y$$

5. Let's next look at the work done by an elastic spring force. When stretched, a linear elastic spring develops a force of magnitude $F_s = ks$, where k is the spring stiffness and s is the displacement of the spring from its unstretched position. Substituting this force expression into the general work equation results in the work of the spring force moving from position s_1 to position s_2 as $[1/2\ k\ (s_2)^2 – 1/2\ k\ (s_1)^2]$. If a particle is attached to the spring, the force F_s exerted on the particle is opposite to that exerted on the spring. Thus, the work done by the spring force will be negative or $U_{1\text{-}2} = –[\ 1/2\ k\ (s_2)^2 – 1/2\ k\ (s_1)^2\ ]$. (Keep in mind that s_1 and s_2 are both displacements from the unstretched position).



Work done by a spring force on an attached particle

6. Important notes to avoid errors in spring force calculations. These equations apply to linear elastic springs only, F = ks. Both s_1 and s_2 are displacements from the unstretched position of the spring, not the absolute positions. The basis for positive or negative work is always whether the spring force acts with or against the direction of motion of the object.

$$U_{1\text{-}2} = –[\ 1/2\ k\ (s_2)^2 – 1/2\ k\ (s_1)^2\ ]$$

7. The principle of work and energy is just the equation of motion written rearranged and written with different variables. Defining $1/2mv^2$ as the kinetic energy of the particle, the principle of work and energy states that the kinetic energy of the particle at position 2 may be obtained by adding the kinetic energy at position 1 to the work done during the displacement from position 1 to position 2 by all the forces acting on the particle.

$$\sum U_{1\text{-}2} = 1/2\ m\ (v_2)^2 - 1/2\ m\ (v_1)^2 \quad \text{or} \quad T_1 + \sum U_{1\text{-}2} = T_2$$

8. Note that the principle of work and energy results in a <u>scalar equation</u>. It can also be applied to a system of particles by summing the kinetic energies of all particles in the system and the work produced from forces acting on the system together.

$$T_1 + \sum U_{1\text{-}2} = T_2$$

9. The advantage of the principle of work and energy is that it allows the direct determination of **velocities and displacements**, rather than solely acceleration from the original form of the equation of motion. Therefore, when problems consist of **forces, velocities, and displacements**, the principle of work and energy can be a more direct method to solve the problem.

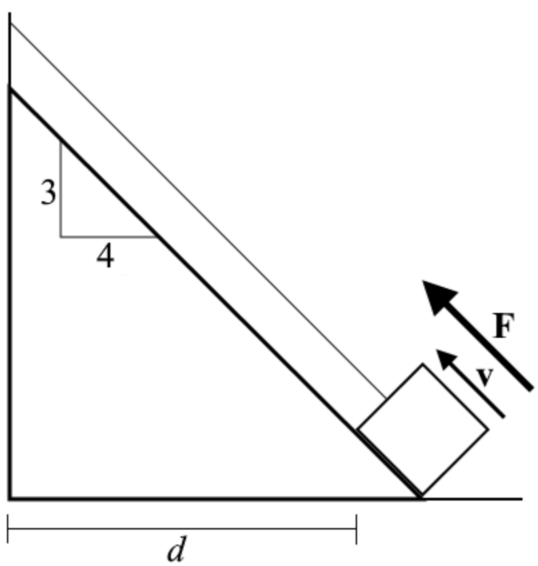
Example 1

The 80 lb block is pulled up the smooth incline starting from rest. Once the block is pulled it moves at a velocity of 7 m/s over 15m. What is the force being exerted on the block?

Given: Weight, Initial Velocity, Final Velocity, Distance

Find: Force exerted on the block

Plan: Apply the Principle of Work and Energy to find the force exerted on the block.

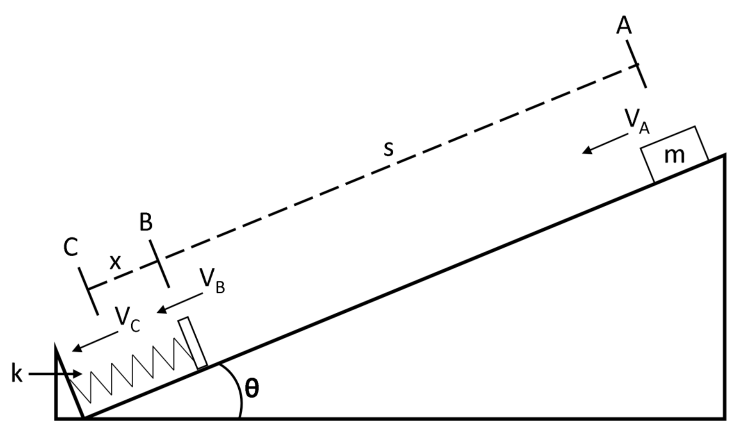


Example 2

Given: A 20-kg block at point A slides 8 meters down the 30° frictionless incline to point B.

Find: The initial velocity V_A, if the block momentarily stops at point C when the spring (k=510 N/m) is compressed 2 meters.

Plan: Draw FBD for block and find net force acting on the block down the ramp. Knowing forces, displacement, and spring compression, find the initial velocity using the principle of work and energy.

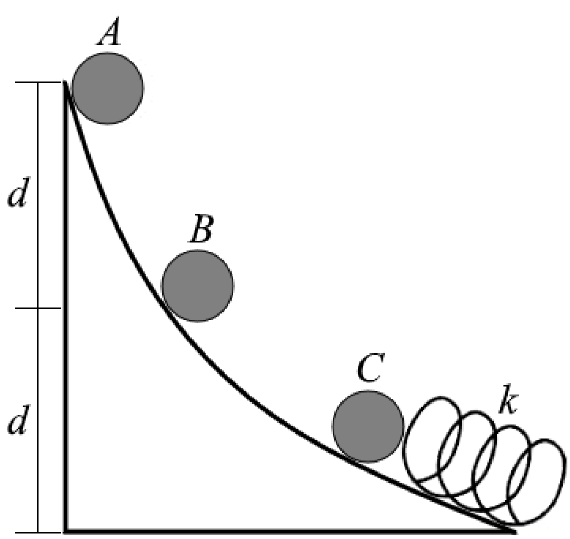


Lesson 12 Group Work

A 30 lb ball with an initial velocity of 5m/s is shot down a smooth slide with a spring in equilibrium at the of the slide with a spring constant of k= 500 N/m^2 . Find the velocity of the ball at point B, then find the maximum compression of the spring. Distance d = 5m.

Given: Weight, Initial Velocity, Spring Constant

Find: V_B , the normal force at B, and maximum compression of spring.



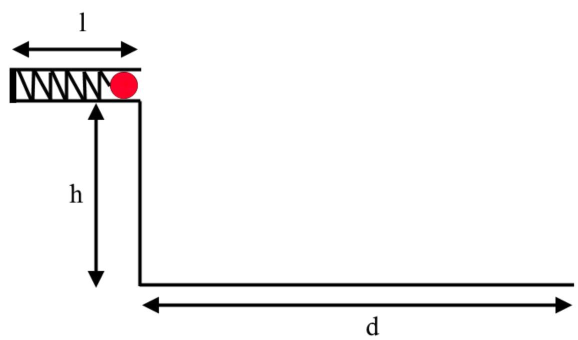
1. Draw the Free Body Diagram of the Ball at A,B, and C

2. Find the Velocity at B

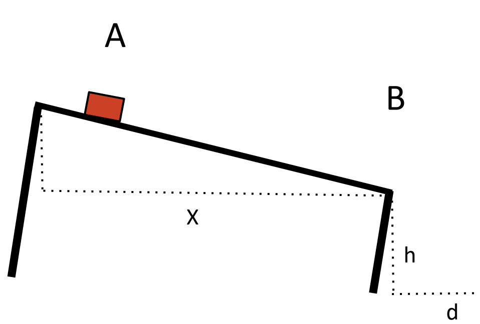
3. Find the maximum compression of the spring at C

Homework Assignment #12

1. A ball of mass m = 5 kg is sitting in a tube of length l = 1m. The spring that the ball is laying against is pulled back to 1/3l and has a spring constant of k = 750 N/m^2 .The ball is then shot out of the tube h=10m above the ground. Find the velocity of the ball leaving the tube and the distance d that the ball traveled.



2. A 5m long table with a frictionless top breaks a leg on the right side causing it to slope. A brick of mass 2-kg resting on the table has a velocity of 3m/s at point A. Find the Velocity at point B and calculate the distance d where the brick will hit the ground if X = 4m and h = 2m.



Power

Lesson 13

1. A force acting over a distance produces work. If a 1 kg object is lifted a distance of 1 meter, the same amount of work is done whether it is lifted in 1 second or 10 seconds. However, there is physical significance in how fast work is being done. Engines, motors and machines are designed to produce work at specified speeds, which introduces **power - the rate at which work is done.**

2. **Power** is useful in determining the time taken and efficiency in numerous applications and is sometimes a more important criterion than the actual amount of work performed. The **speed** at which a truck can climb a hill depends upon the power output of the engine. The motor that raises and lowers an elevator need to have enough power to lift a fully loaded elevator at the desired speed.

3. Power is defined as **the rate at which work is performed**; therefore, the average power done over a certain time interval can be expressed as the derivative of work with respect to time. Expressing work as a vector, $\mathbf{F} \cdot d\mathbf{r}$, allows power to be expressed as $\mathbf{F} \cdot \mathbf{v}$. As a result, power is a scalar defined as the product of the force and velocity components acting in the same direction.

$$P = dU/dt = (\mathbf{F} \cdot d\mathbf{r})/dt = \mathbf{F} \cdot \mathbf{v}$$

$$P = F \cdot v \cdot \cos\theta$$

4. Power is expressed in units of Work/Time or Force • Velocity. In the SI system, 1 Watt is equal to 1 Nm/s. In English units, power is measured in ft-lb/s or more commonly in horsepower.

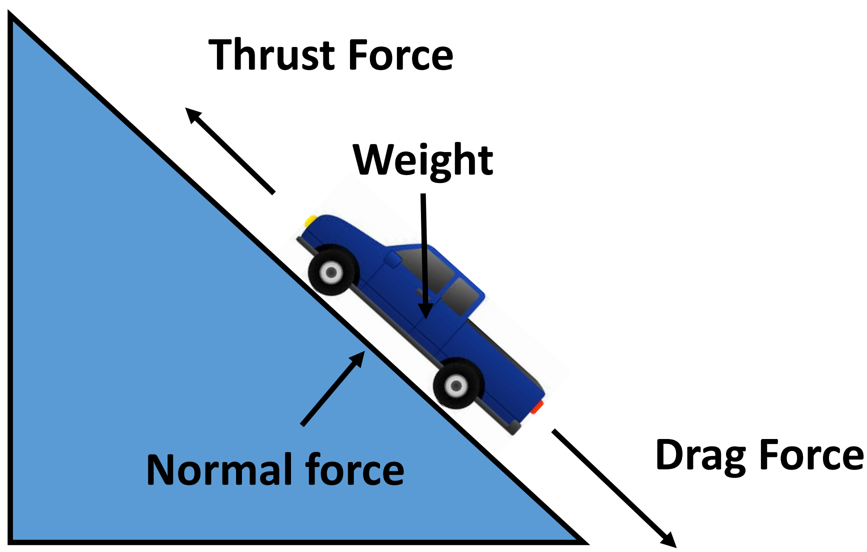
$$1 \text{ hp} = 550 \text{ ft}\cdot\text{-lb/s}$$

5. The ratio of output power to input power is called the **efficiency** of the machine, and is less than one because there will always be losses due to friction. This means that we will never get the same amount of power out of a machine as we put into it.

$$\varepsilon = (\text{energy output}) / (\text{energy input})$$

6. When solving problems involving power and efficiency, knowing the efficiency ratio is helpful in determining input power. After computing force and velocity, which correspond to output power, the efficiency ratio can be applied to solve for input power.

7. A common type of power problem requires an understanding of the forces operating on a moving vehicle. The figure below illustrates the primary forces present on a moving car. The thrust is the force resulting from the engine turning the wheels and generating a friction force which pushes the car forward. The drag force results from wind resistance, and is often a function of velocity squared. The other forces are the same as on a stationary vehicle.



Forces present on a moving car

Procedure for Analysis

1. Find the **resultant external force** acting on the body . It may be necessary to draw a free-body diagram to determine the forces causing motion.

2. Determine the **velocity** of the body to which the force is applied. Energy methods or the equation of motion and appropriate kinematic relations may be necessary.

3. Determine the **power supplied to the body** by multiplying the force magnitude by the velocity component acting in the direction of **F** ($P = F\ v \cos\theta$).

4. In some cases, power may be found by calculating the **work done per unit of time** ($P = dU/dt$).

5. If the **mechanical efficiency** of a machine is known, either the power input or output can be determined.

Example 1

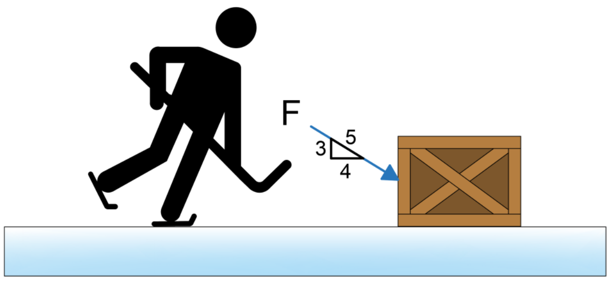
Given: : During conditioning practice, a runner with a mass of 94 kg runs up the stairs of a football stadium. The runner travels a vertical distance of 10.3 meters in 7.24 seconds.

Find: The power the runner generates.

Plan:

1) Draw a free body diagram of the runner.
2) Find the force the runner exerts on the stairs
3) Determine the output power required for this motion.

Example 2



Given: A hockey player pushes a 5-kg box of pucks across the ice rink. Assume friction is negligible and the player/crate starts from rest.

Find: Power generated by a force F =7N when t = 6 s.

Plan:

1) Determine the acceleration of the crate by using the equation of motion in the x-direction.

2) Use this acceleration with kinematics to solve for the crate's velocity at 6 s.

3) Apply the equation of power to solve.

Lesson 13 Group Work

Given: A 65-kg cyclist plans to accelerate uniformly from rest up to a speed of 9 m/s over 18 seconds as rides up the hill. The bicycle's mechanical efficiency $\varepsilon = 0.95$.

The slope of the hill is X = 9ft, Y = 1 ft.

Find: The power the cyclist must provide to the bicycle to achieve this acceleration.

Plan: Calculate the required acceleration

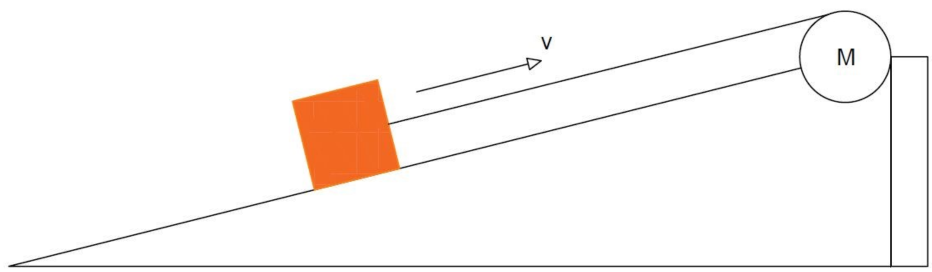
Sum forces to find the magnitude of thrust force

Calculate corresponding power output

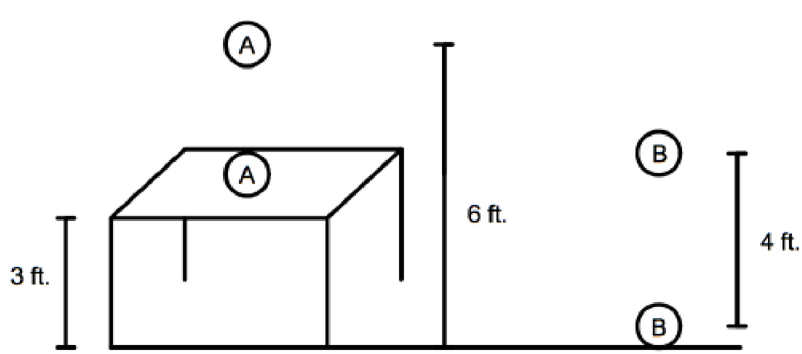
Calculate the required power input

Homework Assignment # 13

1) A motor pulls a 5-g block up an incline with a kinetic friction coefficient of 0.25 at a constant velocity of 2.357 m/s. What is the power supplied to the motor if it takes 3 seconds for the block to reach the top of the incline?



2) A girl lifts a 30-lb weight B from the ground to a height of 4 feet in 2 seconds. Her friend lifts a 45-lb weight A from a tabletop 3 feet off the ground to a height of 6 feet above the ground in 3 seconds. Which person generated more power? How high could that person lift a 60-lb weight if they did so in the same amount of time and exerted the same amount of power as they did with their first weight?

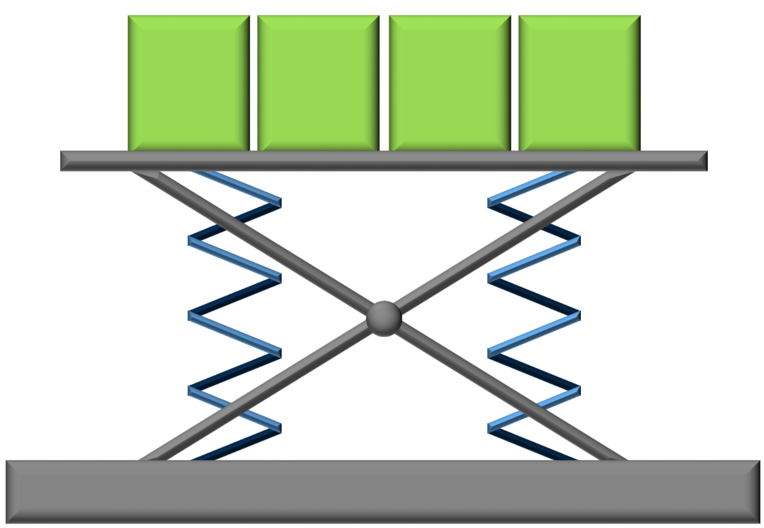


Conservation of Energy

Lesson 14

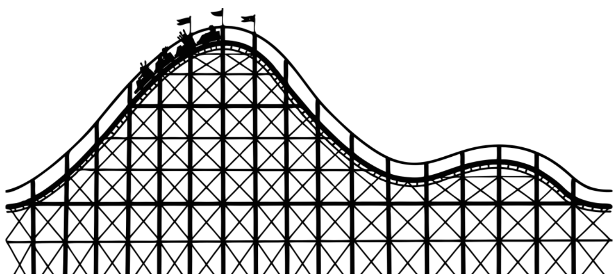
1. In previous lessons covering the principles of work and energy, we never mentioned the word potential energy. In this lesson, we will study problems with certain forces that allow a different energy approach, the principle of the **conservation of energy.** This requires us to study potential energy and the types of conservative forces associated with it.

2. The figure shown below is an example of how energy is conserved when transformed from one type of energy to another. In this case, the weight of the blocks have gravitational potential energy that transforms into elastic potential energy in the springs as they compress. Throughout this process, energy is conserved.



Blocks supported on a spring platform

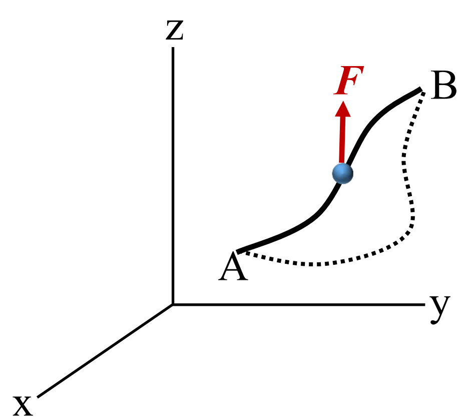
3. The roller coaster is another illustration of the interplay between potential energy and kinetic energy. At the top of a hill, the car has gravitational potential energy, which is converted to kinetic energy as it moves down along the track. As the car moves back up, its kinetic energy converts back to gravitational potential. If we ignore friction, energy is conserved in both conversions. This is true because only gravity, a conservative force, acts on the roller coaster car.



Energy transformation of a roller coaster

4. We can apply the principle of conservation of energy to systems acted upon **only by conservative forces**. A conservative force does work that is independent of a particle's path and only depends on the initial and final positions. This is different from externally-applied forces and friction forces, which are path-dependent, and therefore non-conservative. In mechanical systems, **weight** (gravity) forces and **spring** (elastic) forces are common examples of conservative forces. The work that weight and spring forces do are path independent, functions of position, and are recoverable (conserved).

$$\oint \boldsymbol{F} \cdot d\boldsymbol{r} = 0$$



5. We can define potential energy functions associated with conservative forces. A particle's potential energy represents the potential work than can be done on the particle. This is a function of position (for example, a ball gains greater potential energy as it is held higher from the ground).

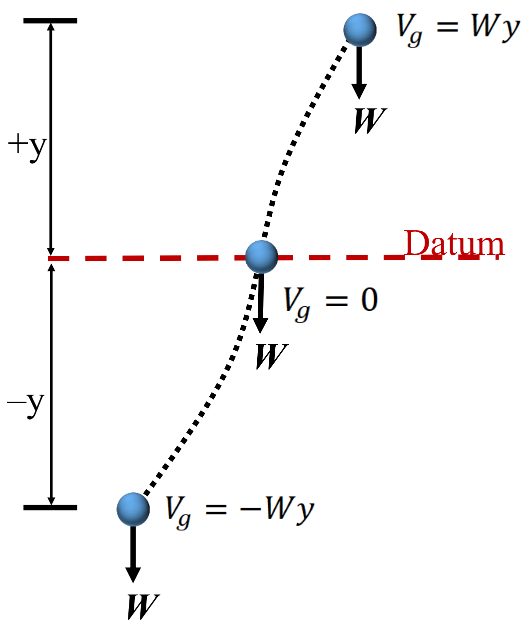
The total potential energy of a mechanical system is the sum of gravitational potential energy and elastic potential energy:

$$\Sigma V = V_{gravity} + V_{springs}$$

6. In gravitational potential energy, the weight of a particle suspended in the air has the capacity to do work depending on its position. The energy is calculated as the **weight** times the **vertical distance** measured from a selected datum, or:

$$V_g = \pm W y$$

This is shown in the figure below:



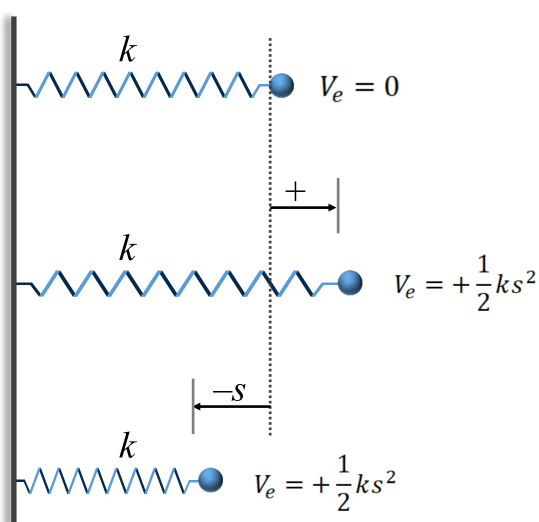
Gravitational potential energy

Notice when the particle is above the datum, potential energy is positive, and when it is below the datum, potential energy is negative.

7. For elastic potential energy, a particle attached to a compressed spring has the capacity to do work. Thus, this amount of potential work determines the amount of energy stored in a spring. We can calculate this work by using the spring force acting through a distance *s* as *1/2 ks*^{2}, where *s* represents the distance between the particle's location and the un-stretched length of the spring. Therefore, the spring force is:

$$V_e = \frac{1}{2}ks^2$$

Again, this concept is demonstrated in a figure:



Elastic potential energy

Notice how the elastic potential energy is always positive regardless of whether the spring is compressed or elongated, as a negative *s* value is simply squared.

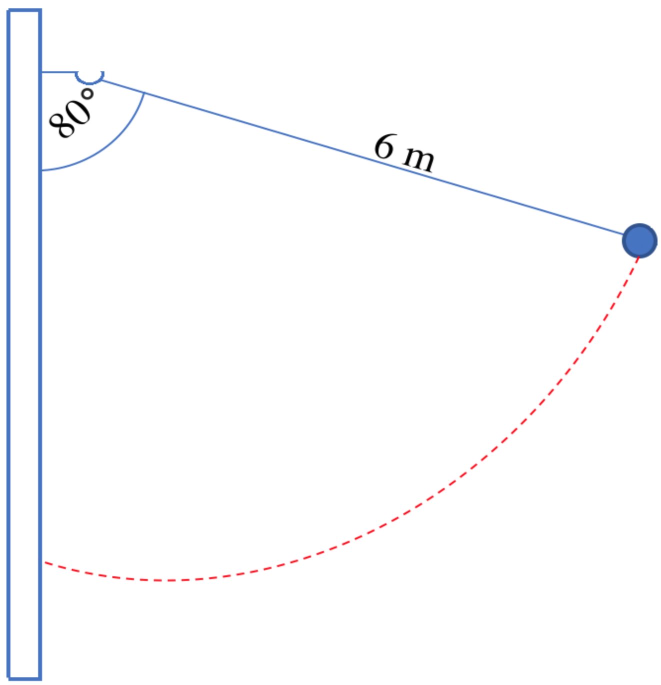
9. Lastly, when a particle is acted upon by a set of conservative forces, total energy is conserved. Here we can apply the principle of conservation of energy, which states that the sum of the potential and kinetic energies at an initial position must equal to the sum at the final position. This gives us the equation:

$$\mathbf{T_1} + \Sigma\mathbf{V_1} = \mathbf{T_2} + \Sigma\mathbf{V_2} = a\ constant$$

Example 1

Given: A 3 kg-ball on a string is connected to a hook in a wall. The ball is dropped from its initial position shown.

Find: Calculate the velocity of the ball and the tension in the string just before it hits the wall.



Plan:

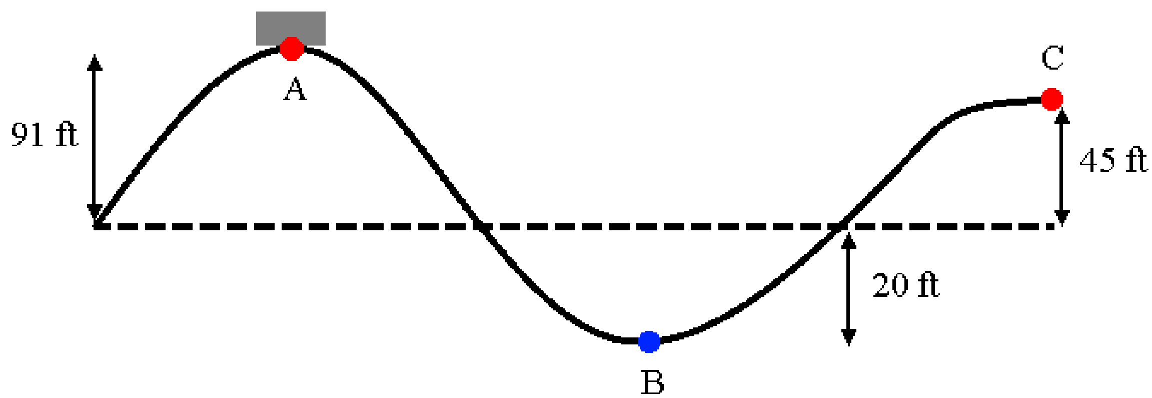
1) Apply conservation of energy to determine the velocity
2) Use n-t equation of motion to determine tension

Example 2

Given: A car of weight 3000 lbs moves along the path shown at a speed of 30 mi/hr. At the top of hill A, the engine is disengaged and the car coasts down the hill. The point B is 20 ft below the datum. Hill C is 45 ft tall.

Find: The velocity of the car at point B and C (friction is negligible).

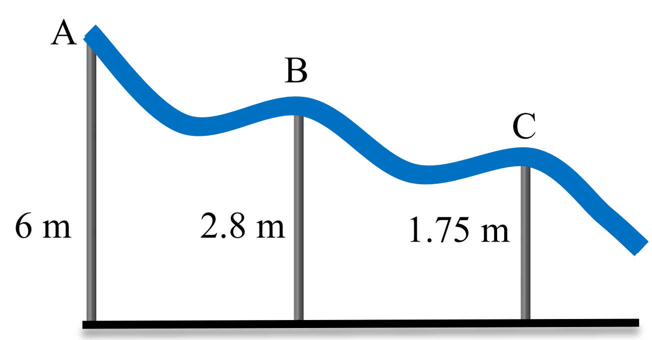
Plan: Draw a diagram. Use Conservation of Energy from one point to another. Then, solve for velocity. (Remember to use units constantly!)



Lesson 14 Group Work

Given: A block of mass m = 10 kg is released from rest at point **A** and slides on the frictionless track.

Find: (a) Determine the block speed at points **B** and **C**. (b) Also, find the net work done by the gravitational force on the block as it moves from point **A** to point **C**.



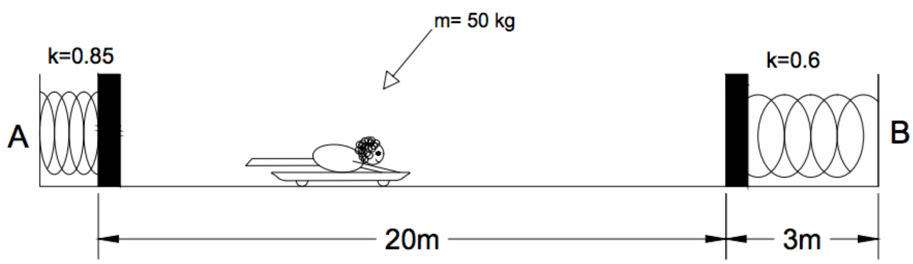
Plan:

Use Conservation of Energy equation to find v_B and v_C :

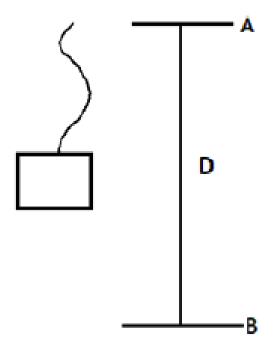
Use $W_{A \to C} = \Delta T$ to find Work:

Homework Assignment # 14

1. A man lying on a skateboard has a total mass of 50 kg. He is loaded on a spring that is compressed 3 m with a spring constant k_A = 0.85. Another spring with a spring constant k_B = 0.6 is placed 20 m away that is supposed to "cushion" the impact and spring the man on the skateboard back with some velocity. Due to friction the man loses 0.75 J/m of energy. Determine the velocity of the man just before contact with the spring and the distance the spring is compressed.



2. A 20-kg block is dropped from a height of D = 75m with an initial downward speed of 2 m/s. Determine the required unstretched length of the elastic cord to which it is attached in order that it stops momentarily just above the ground. The stiffness of the elastic cord is k = 2kN/m. Neglect the size of the block.

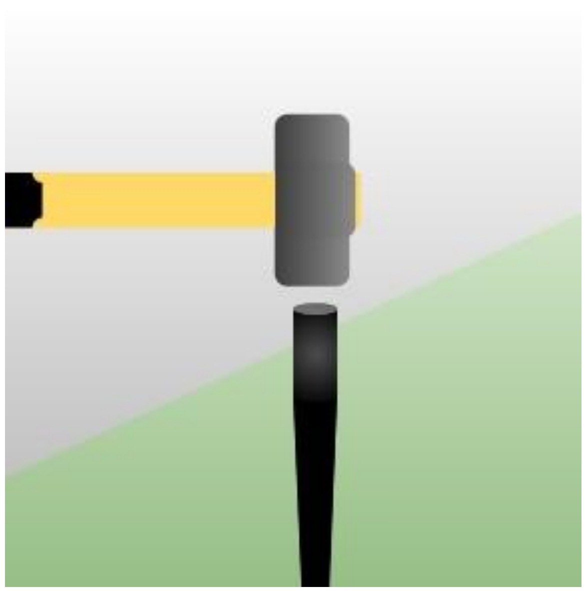


Principle of Impulse and Momentum

Lesson 15

1. The impulse-momentum method will be the last of the three approaches used to analyze particle kinetic problems. In analyzing particle kinetics, we have looked at the direct application of the equation of motion and one reconfigured form of the equation of motion: the principle of work and energy. We will **reconfigure the equation of motion** again to derive the **principle of impulse-momentum**. The impulse-momentum method will evaluate forces **applied to the particle over time**, and how the particle's motion is affected. Impulse-momentum is applied to problems involving force, velocity, and time.

2. The picture shows a stake driven into the ground by a sledgehammer. From the stake's perspective, it receives a large magnitude of force applied over a short period of time, which changes its velocity from zero to some magnitude downwards.



Hammer driving a stake into the ground

3. Rewriting acceleration in the equation of motion as the change in velocity with respect to time, separating variables, and integrating the force results in the principle of impulse-momentum. We call **mv** the **momentum** of the particle and **∫F dt** the **impulse** of the force **F**. The principle of impulse-momentum represents the change in momentum of the particle as a result of impulse forces applied to the particle over a period of time.

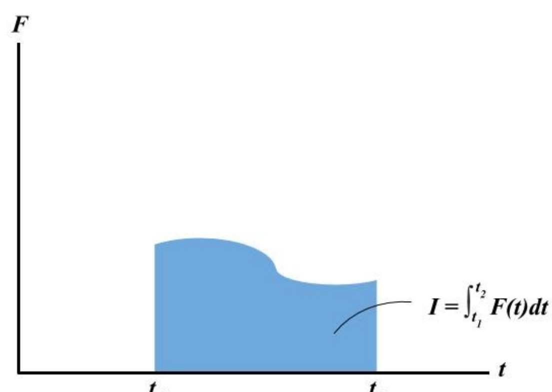
$$\int F\,dt = \int m\,dv$$

$$\underbrace{\sum \int_{t_1}^{t_2} \boldsymbol{F}\,\mathrm{dt}}_{\text{The impulse applied}} = \mathrm{m} \int_{\boldsymbol{v}_1}^{\boldsymbol{v}_2} \mathrm{d}\boldsymbol{v} = \underbrace{\mathrm{m}\boldsymbol{v}_2 - \mathrm{m}\boldsymbol{v}_1}_{\text{The change in momentum}}$$

$$\boxed{\mathrm{m}\boldsymbol{v}_1 + \sum \int_{t_1}^{t_2} \boldsymbol{F}\,\mathrm{dt} = \mathrm{m}\boldsymbol{v}_2}$$

The impulse may be determined by integration, either direct or graphical. Graphically, it can be represented by the area under the force versus time curve. If $\boldsymbol{F}$ is constant, then

Impulse *(I)* = $\boldsymbol{F}\,(t_2 - t_1)$.



Variable Force

4. Momentum is the resultant vector of mass times velocity in the direction of the velocity. Intuitively, momentum may be thought of as the force of movement. The more momentum an object has, the more difficult it will be to stop. Impulse is a measure of the accumulated effect of a force over some interval of time. Impulse is a vector in the same direction as the force. The sum of all the force impulses acting on the particle is what appears in the equation. We may determine the impulse by integrating the force with respect to time, either by direct integration, or graphical integration.

5. The impulse of the resultant force acting on a particle is equal to the change in the particle's momentum.

$$\sum \int_{t_1}^{t_2} \boldsymbol{F}\, \mathrm{dt} = \mathrm{m}\boldsymbol{v}_2 - \mathrm{m}\boldsymbol{v}_1$$

Applying the principle of impulse-momentum involves drawing three diagrams: the initial momentum, the impulse, and the final momentum diagrams.

Initial momentum diagram	+	Impulse diagram	=	Final momentum diagram

6. Since the principle of impulse-momentum is a vector equation, we may extract **three scalar equations** from it with respect to a particular coordinate system.

$$m(v_x)_1 + \sum \int_{t_1}^{t_2} F_x \, dt = m(v_x)_2$$

$$m(v_y)_1 + \sum \int_{t_1}^{t_2} F \, dt = m(v_y)_2$$

$$m(v_z)_1 + \sum \int_{t_1}^{t_2} F_z \, dt = m(v_z)_2$$

7. **Steps** involved in solving problems with the principle of impulse-momentum:

1) Establish the x, y, z coordinate system.

2) Draw the particle's free body diagram and establish the direction of the particle's initial and final velocities by drawing the impulse and momentum diagrams for the particle. Show the linear momenta and force impulse vectors.

3) Resolve the force and velocity (or impulse and momentum) vectors into their x, y, z components and apply the principle of linear impulse and momentum using its scalar form.

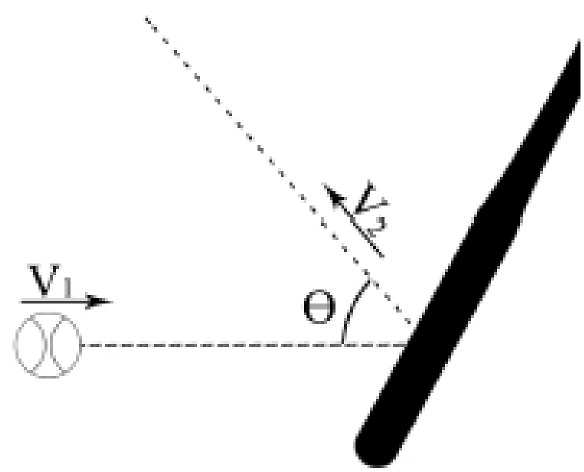
4) Forces as functions of time must be integrated to obtain impulses. If a force is constant, its impulse is the product of the force's magnitude and time interval over which it acts.

Example 1

Given: A baseball weighing 0.2 lbs has a velocity of $v_1 = 35$ ft/s when it strikes the baseball bat and leaves the bat with a velocity of $v_2 = 22$ ft/s at an angle $\theta = 40^0$ with the horizontal.

Find: The magnitude of the impulsive force responsible for changed velocity.

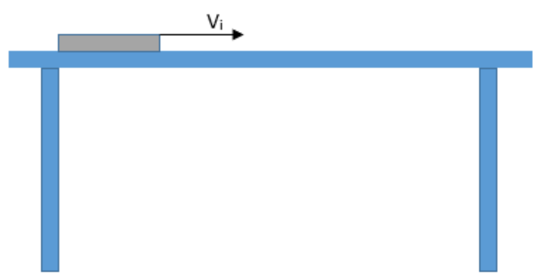
Plan: Use the Principle of Impulse-Momentum



Example 2

Given: A 3.5 kg book is slid across the desk such that its initial velocity is 2 m/s. The coefficient of kinetic friction between the book and the desk is 0.05.

Find: a) the speed of the block after 2 seconds and (b) the time at what time the book comes to rest.



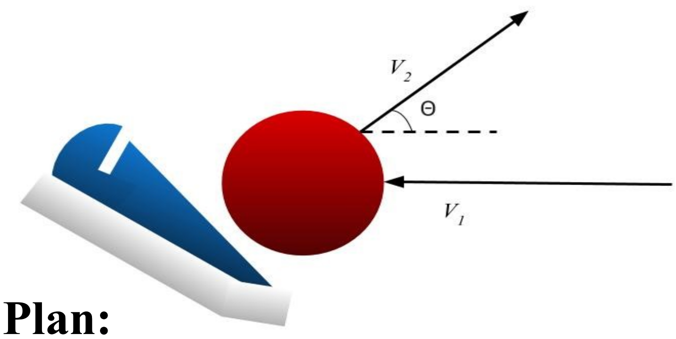
Plan:
Apply Principle of Impulse-Momentum to determine v after 2 seconds

Re-apply Principle of Impulse-Momentum to determine time to reach zero velocity

Lesson 15 Group Work

Given: A player kicks a 0.5-kg kickball moving 5 m/s on the ground. It leaves the player's foot with a speed of 20 m/s at an angle of 30 degrees. The foot and ball are in contact for 0.02 seconds.

Find: The magnitude and direction of the average force exerted by the player's foot on the ball.



Plan:

Apply impulse momentum equation in x-direction:

Apply impulse momentum equation in y-direction:

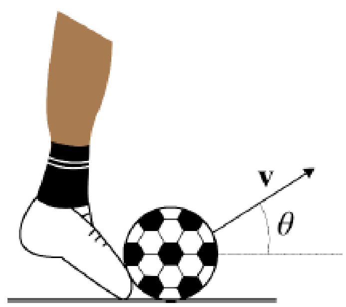
Solve for the magnitude of the average force to find angle:

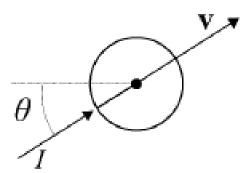
Plug in angle to find the magnitude of the average force:

Lesson 15 Group Work [2]

Given: A soccer star kicks an m-kg ball at an angle of θ, such that it eventually hits the ground at the same elevation, d m away. Use m = 17 kg, d = 15 m, and $\theta = 55^0$.

Find: The impulse of the player's foot on the ball. Neglect the impulse created by the ball's weight while it's being kicked.





Plan: Draw FBD of the ball

Write equation for velocity in the x-direction

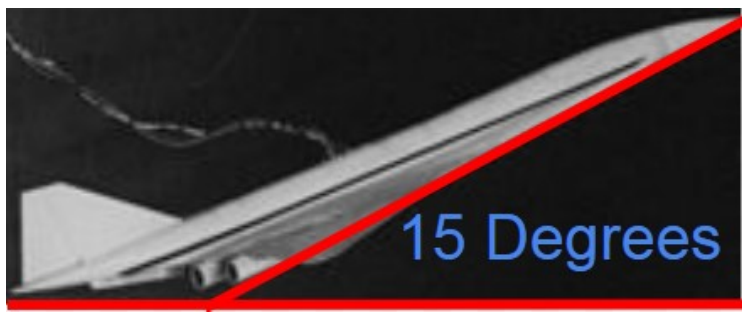
Write equation for velocity in the y-direction

Solve for velocity by relating t to both directions

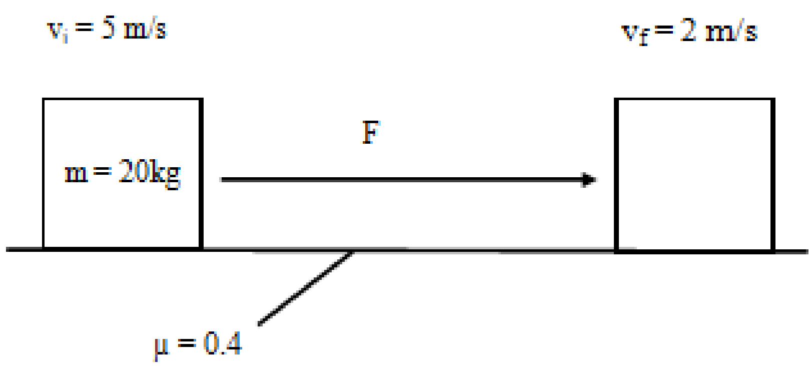
Use Impulse-Momentum to solve for the impulse force

Homework Assignment # 15

1. The Concorde was a supersonic transport plane weighing 412,000 lbs on takeoff and used 4 engines with 38,050 lbs of thrust each. If the plane took off with an angle of 15° at a speed of 400 mph and continued this course for 10 seconds, what would the final velocity components be? Lift is normal to the wings and is enough to cancel out the weight.

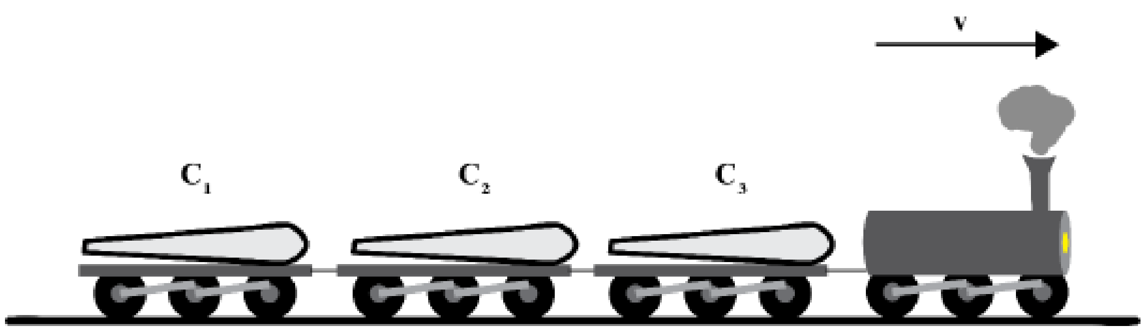


2. A block of mass m is sliding with an initial velocity v_i along a surface with friction coefficient μ. Find the average applied force acting on the block if it reaches velocity v_f after 4 seconds.



Homework Assignment # 15

3. A train is made up of a 60Mg engine and three cars of wind turbine blades, each having a mass of 40Mg. The train begins in motion at 3km/hr after pulling out from the railyard. The train then accelerates uniformly and achieves a top speed of 50 km/hr after one minute. The only force applied is by the wheels on the engine in contact with the rails, as each of the car's wheels roll without friction. Find the force needed to accelerate the train to speed in 60 seconds.



Conservation of Momentum

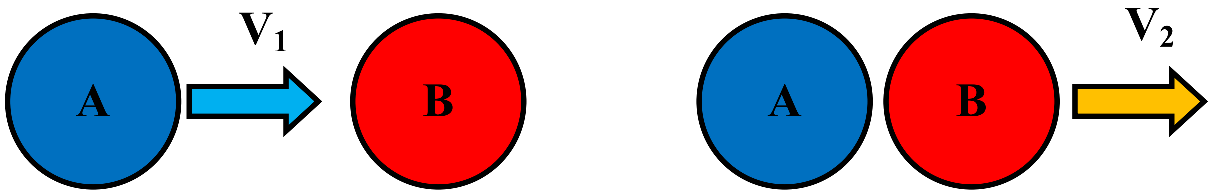
Lesson 16

1) The principle of linear impulse and momentum can also be applied to a **system of particles**, rather than one single particle. When applied to a system of particles, it is important to recognize the significance of external impulse forces applied to the system. If the sum of all external forces acting on a system of particles is equal to zero, then the momentum for the system is said to be **conserved** and can be written as such:

$$\sum \mathbf{m_1 v_1} = \sum \mathbf{m_2 v_2}$$

This equivalency indicates that the total linear momentum for a system of particles does not change over a time interval unless an **external impulse force** is applied to the system.

2) Particle collision and explosion are common examples of systems without external impulse forces; therefore, **conservation of momentum** can be applied to both cases.



Before

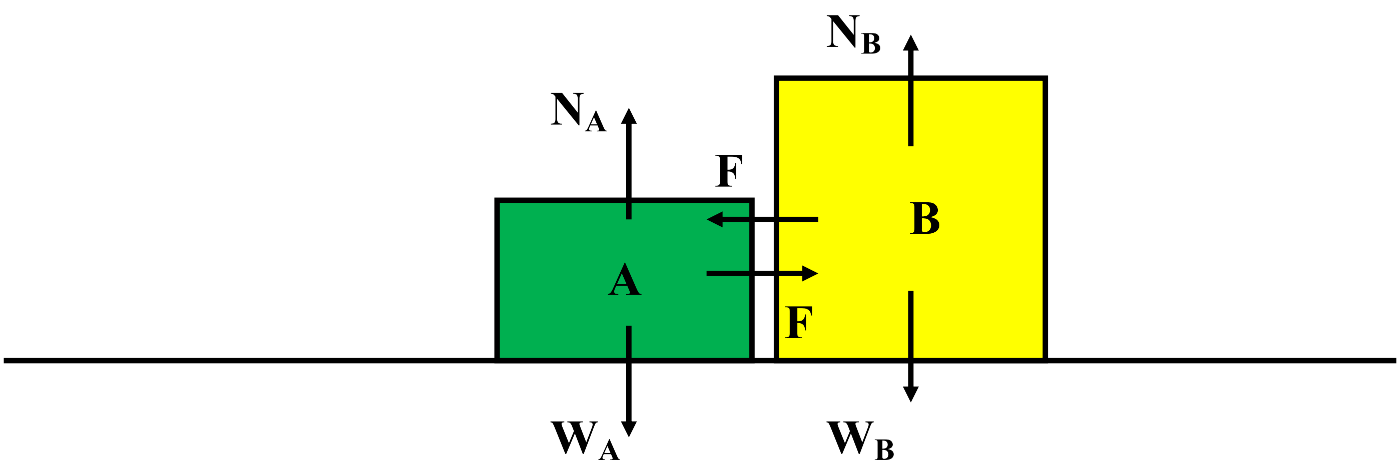
After

3) Only external impulse forces need to be considered when applying principles of linear impulse and momentum because internal impulses will always occur in **equal and opposite pairs**, effectively cancelling out. These forces, which cause no change in momentum, are referred to as **non-impulsive forces** and can be neglected in calculating momentum. However, forces which do impact momentum are termed **impulsive forces** and must be accounted for in computation. The full momentum equation for a system of particles can be written including impulse forces:

$$\sum \mathbf{m_1 v_1} + \sum \int \mathbf{F_t \, dt} = \sum \mathbf{m_2 v_2}$$

Performing with integration over the specific **time interval**, t_1 to t_2, in which the impulse forces are acting will result in the summation of external impulse forces. Even when the total momentum of the system remains constant, the momentum of each individual particle can and often will change.

4) Impulsive forces generally occur from an explosion or one particle colliding with another, while non-impulsive forces are generally other negligibly small or offsetting impulse forces. For example, in the moment these two blocks collide, all forces have equal and opposite counterparts, so all forces are considered non-impulsive.



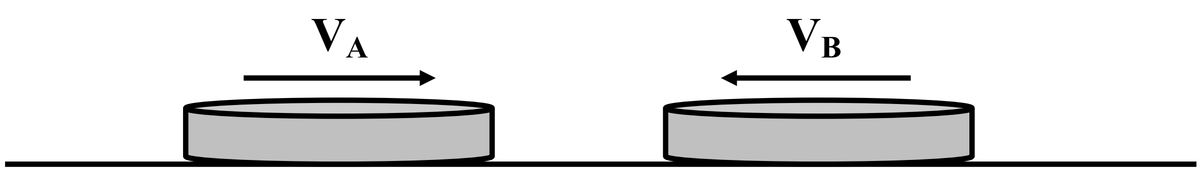
Impulse forces during a collision

Example 1

Given: Two hockey pucks collide on the ice. Hockey puck A has a mass of 212 g and hockey puck B has a mass of 156 g. The velocity of A before the collision is $V_A = 52$ m/s. Post collision, puck A has a velocity of 46 m/s and puck B has a velocity of 67 m/s.

Find: The velocity V_B which puck B collides with puck A if the pucks rebound such that puck A moves to the left and puck B moves to the right. Also determine the time the collision occurs in if the impulsive force is 6.36 N.

Plan: Solve for the velocity of puck B using conservation of linear momentum for the two pucks. Then solve for the time using the principle of linear impulse and momentum.



Example 2

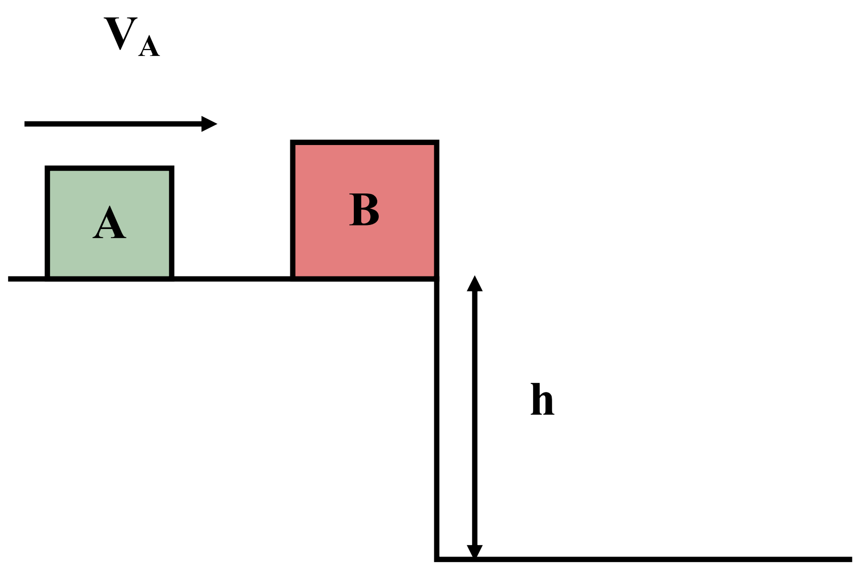
A 15 kg particle and a 10 kg particle are both travelling in the same direction. The 15 kg particle is behind the 10 kg particle. The 15 kg particle is traveling at a velocity of 10 m/s. When they collide, the momentum is conserved and they stick together. What is the common final velocity of the particles?

Lesson 16 Group Work

Given: A 10 kg block A slides towards a stationary block B of mass 15 kg at the top of a drop of height h = 1.75. The approach velocity of A is $V_A = 7.4$ m/s. After the collision the two blocks stick together and fall off the edge towards the ground.

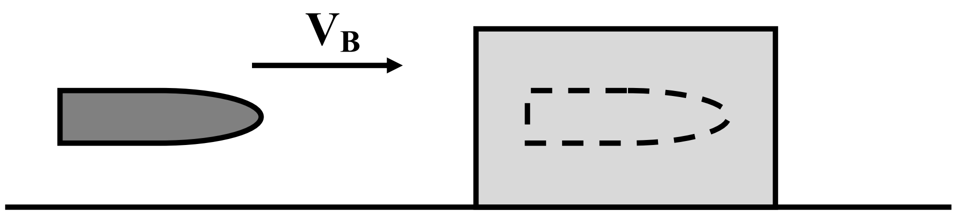
Find: Determine the speed with which the combined blocks hit the ground.

Plan: Use conservation of momentum to solve for the velocity that the block goes off the edge with. Use projectile motion to solve for the speed when the block hits the ground.



Homework Assignment #16

A wood block with the mass of 5 kg sits on a table with a coefficient of friction of $\mu_k = 0.3$. A bullet of mass 8 g hits the block and remains in the block after the collision. The block moves a distance of 0.5 m after the collision. Find the initial velocity of the bullet.

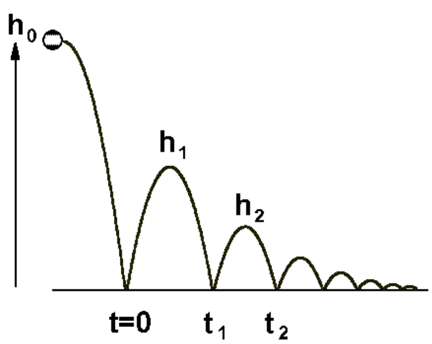


Impact

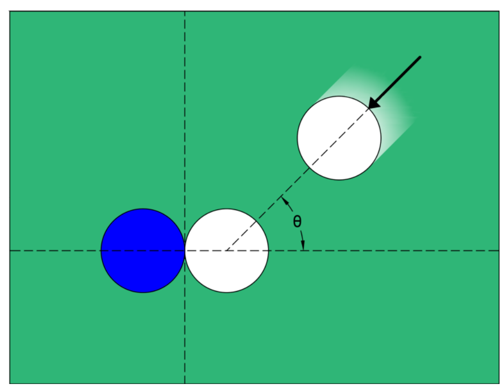
Lesson 17

1) An impact occurs **when two bodies collide** and exert impulse forces on one another. We have already discussed how the principle of **conservation of momentum** is useful in analyzing collision problems.

2) When a ball bounces, it does not return to its original height. The loss in height implies that **some energy is lost** during the collision with the ground.



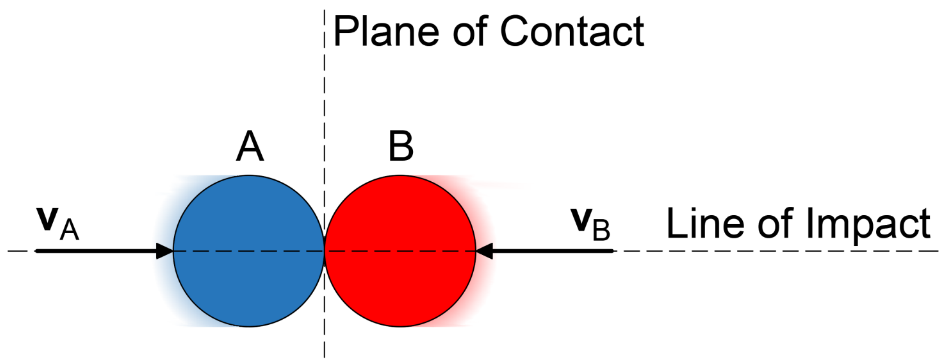
Energy loss



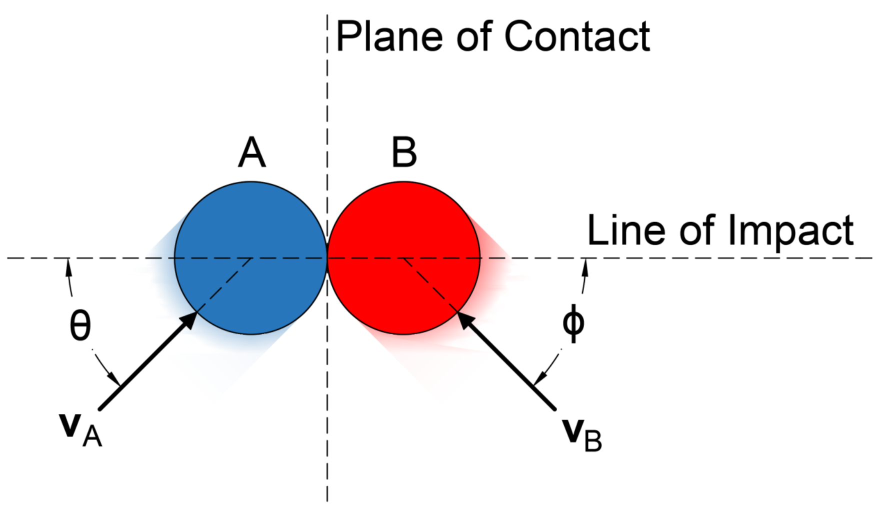
Oblique impact example

3) When one ball is hit head-on by another, the stationary ball will move in the same direction as the ball in motion. But if the collision instead occurs at an angle, the stationary ball will move at an angle. It will be necessary to determine the magnitude and the direction of the impacted ball for either situation.

4) The **line of impact** is represented by a line through the center of mass of both objects and the point of contact. The **plane of contact** is perpendicular to the line of impact and through the point of contact. There are two types of impact problems: **central impact**, when the collision is head on, and **oblique impact**, when the collision is at an angle. During a central impact, both objects move along the line of impact. During an oblique impact, both objects move at an angle to the line of impact.



Central Impact



Oblique Impact

5) In **central impact** problems, the two initial velocities are usually given with the goal of determining the magnitude of the two final velocities. There are therefore **two unknowns**, which can be solved using two equations.

6) The **first equation will be the conservation of system momentum along the line of impact**.

$$(\mathbf{m_A\ v_A})_1 + (\mathbf{m_B\ v_B})_1 = (\mathbf{m_A\ v_A})_2 + (\mathbf{m_B\ v_B})_2$$

The second equation will be based on the principle of impulse-momentum and **will account for the elasticity of the physical collision** and the resulting energy loss.

7) The **coefficient of restitution** is a way to measure the elasticity of a collision. It relates the initial and final velocities of the colliding objects. The coefficient is found by taking the ratio of the relative final velocities to the relative initial velocities.

$$e = \frac{(\mathbf{v_B})_2 - (\mathbf{v_A})_2}{(\mathbf{v_A})_1 - (\mathbf{v_B})_1}$$

8) The value of e will always be between 0 and 1, with 0 representing **perfectly plastic** collisions, and 1 representing **perfectly elastic** collisions. A perfectly plastic collision occurs when two objects stick together after a collision. A perfectly elastic collision occurs when no energy lost during the collision.

9) Using the **conservation of system momentum** and the equation for the **coefficient of restitution along the line of impact**, we can solve for the magnitude of the final velocities of the two objects after impact. With these two velocities, we can **obtain the energy loss as a result of the impact** by finding the difference in system kinetic energy.

$$\Sigma U_{1\text{-}2} = \Sigma T_2 - \Sigma T_1 \quad \text{where } T_i = 1/2\ m_i (v_i)^2$$

10) In an **oblique impact**, there are four unknowns: the magnitudes and directions of the final velocities. After establishing coordinate directions along the line of impact and perpendicular to the line of impact, the following equations can be applied to solve for each of the four **unknowns**:

Along the line of impact (x-axis):

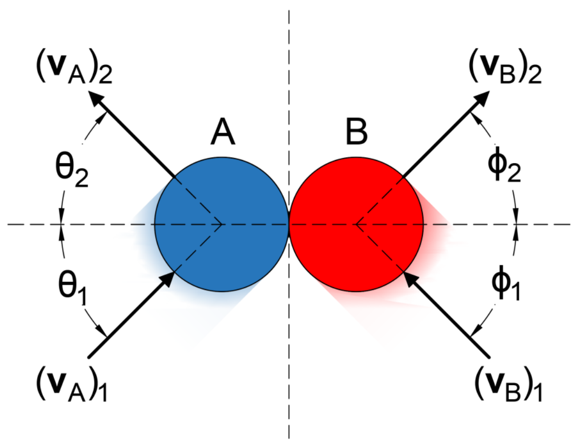
$$\mathbf{m_A(v_{Ax})_1 + m_B(v_{Bx})_1 = m_A(v_{Ax})_2 + m_B(v_{Bx})_2}$$

$$\boldsymbol{e} = \mathbf{[(v_{Bx})_2 - (v_{Ax})_2]/[(v_{Ax})_1 - (v_{Bx})_1]}$$

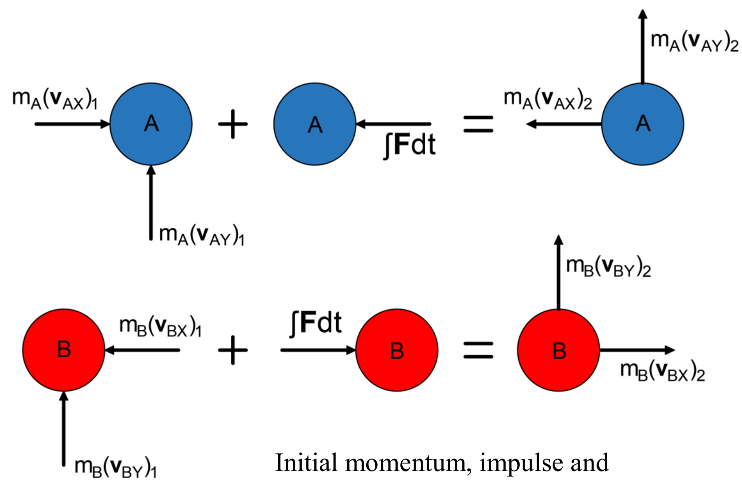
Perpendicular to the line of impact (y-axis):

$$\mathbf{m_A(v_{Ay})_1 = m_A(v_{Ay})_2 \quad and \quad m_B(v_{By})_1 = m_B(v_{By})_2}$$

The two equations **along the line of impact** are the same as used for central impact problems, **conservation of system momentum and the equation for the coefficient of restitution**. Since none of the impulse forces are **perpendicular to the line of impact**, each **particle's momentum in that direction is conserved**, giving us the other two equations.



Oblique impact



Initial momentum, impulse and final momentum diagrams

Steps Summary

1) When given the initial velocities and the coefficient of restitution, the final velocities can be determined.

2) Define the x-y axes. Typically, the x-axis is defined along the line of impact, and the y-axis is in the plane of contact perpendicular to the x-axis.

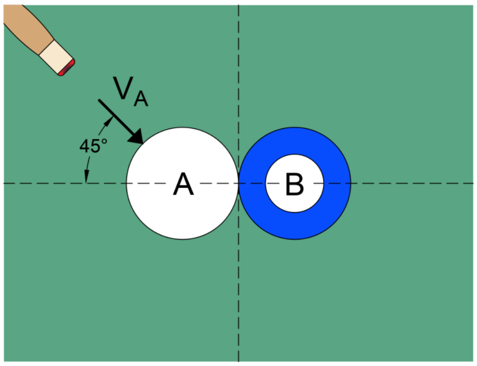
3) For both central and oblique impact problems, the following equations apply along the line of impact (x-dir.):

$\Sigma\, m(v_x)_1 = \Sigma\, m(v_x)_2$ and $e = [(v_{Bx})_2 - (v_{Ax})_2]/[(v_{Ax})_1 - (v_{Bx})_1]$

4) For oblique impact problems, the following equations are also required, applied perpendicular to the line of impact (y-dir.):

$m_A(v_{Ay})_1 = m_A(v_{Ay})_2$ and $m_B(v_{By})_1 = m_B(v_{By})_2$

Example 1



Given: Two pool balls of mass 0.16 kg collide. The cue ball (ball A) has a velocity of 10 m/s while ball B is stationary. The given coefficient of restitution is 0.8.

Find: The x and y components of the final velocity for each ball after the collision.

Plan:

1) Draw a diagram of the collision, defining the x-y axes and line of impact.

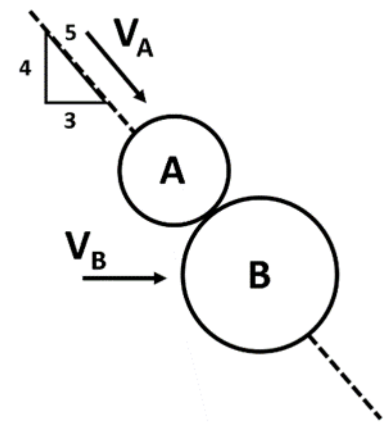
2) Apply conservation of momentum and coefficient of restitution equations in the x-direction to solve for the x-components of final velocity.

3) Use the equations for motion perpendicular to the line of impact to solve for the y-components of final velocity.

Example 2

Given: Ball A and B have masses of 1 kg and 2 kg, respectfully.

Find: Their speeds after impact if the coefficient of restitution is e = 0.6 and their initial speeds are V_A = 8 m/s and V_B = 10 m/s.



Plan:

Use Impact equation to relate $(V_A)_2$ and $(V_B)_2$:

Use restitution equation to relate $(V_A)_2$ and $(V_B)_2$:

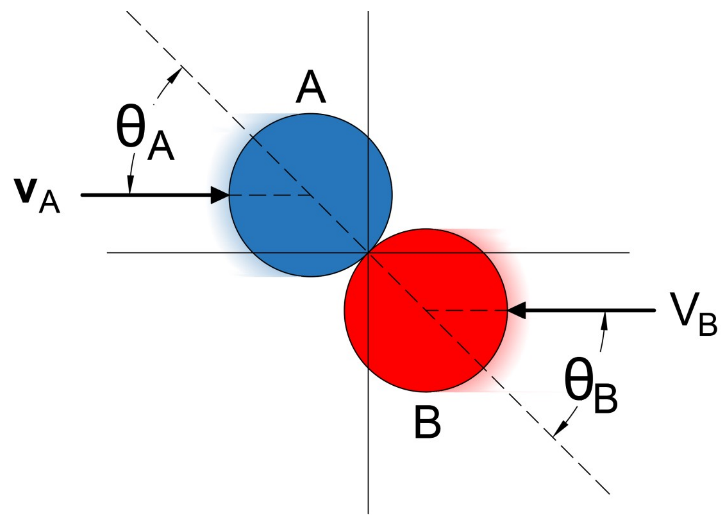
Solve system of equations and solve for $(V_A)_2$ and $(V_B)_2$:

Lesson 17 Group Work

Given: Two smooth disks A and B, both having a mass of 2 kg, collide with the velocities V_A= 5 m/s and V_B= 8 m/s as shown. If the coefficient of restitution for the disks is e = .60

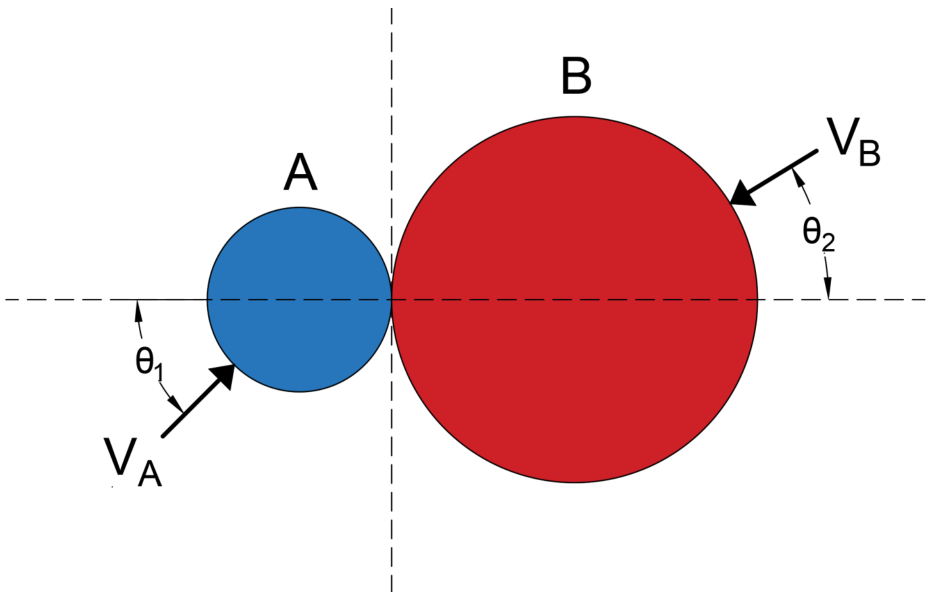
Find: The x and y components of the final velocity of each disk just after the collision when θ = 30°.

Plan:



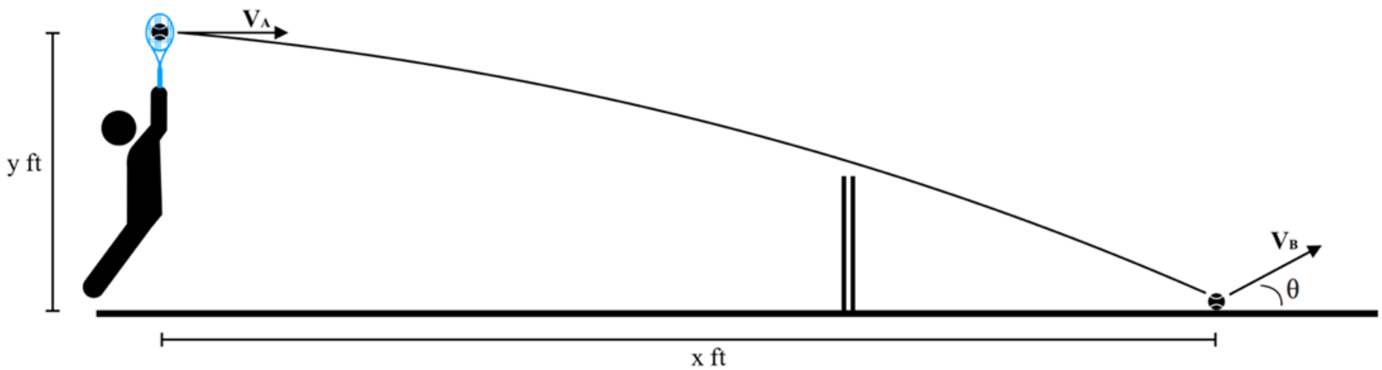
Homework Assignment # 17

1. Two spheres A and B collide against one another. Their masses are 5 kg and 17 kg respectively. Ball A travels 4 m/s with an angle $\theta_1 = 45^o$ from the line of impact. Ball B travels the opposite direction 4 m/s with an angle of $\theta_2 = 35^o$ above the line of impact. The coefficient of restitution for the impact is 0.75. Determine the x and y components of the final velocities for each of the spheres.



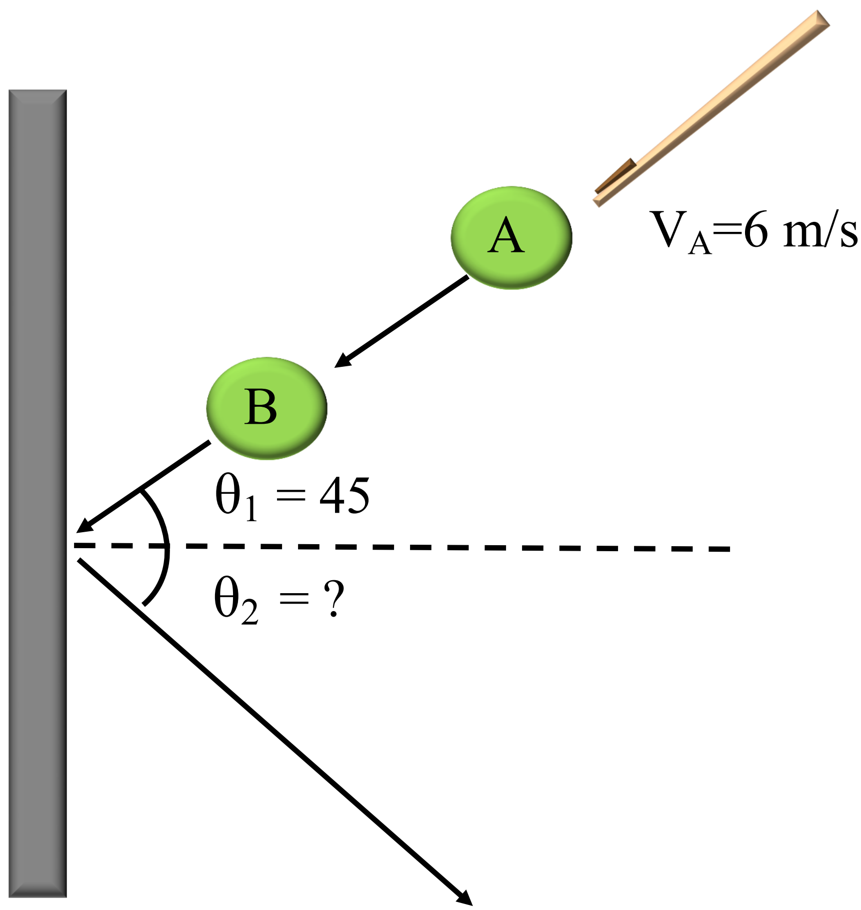
Homework Assignment # 17

2. A person serves a tennis ball in the horizontal direction from point A at a height of y = 9ft. The ball hits the ground at point B, which is x = 60ft horizontally from point A. Assume the ground is smooth and that the angle θ at point B is 15°. Calculate the horizontal velocities of the ball at points A and B, as well as the coefficient of restitution e.



Homework Assignment # 17

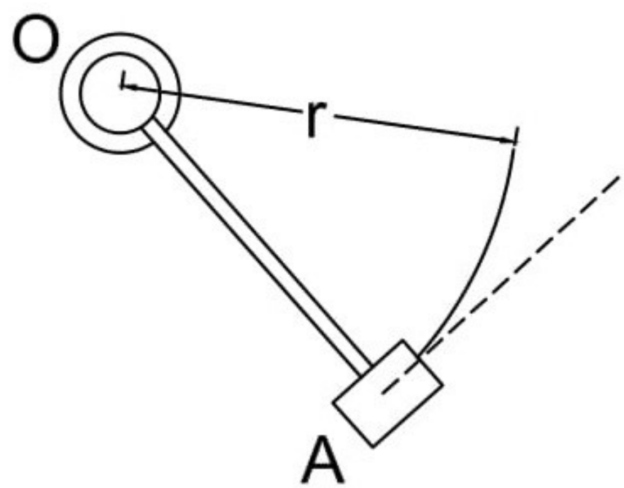
3. Cue ball A is shot with an initial velocity of 6 m/s. After it makes a direct collision with ball B, e =0.9, determine the velocity of ball B and the angle made after it rebounds from point C on the table wall (e =0.7). Both balls have an equal mass of 3 kg.



Angular Impulse and Momentum

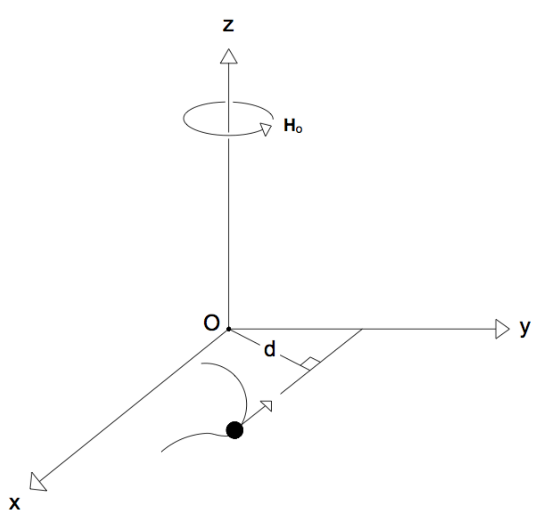
Lesson 18

1) Thus far, we have been studying momentum that occurs along a line of action. In short, it has been *linear momentum*. *However, we will now study momentum that rotates around a point. This type of momentum is referred to as angular momentum.* Angular momentum is the rotational counterpart to linear momentum. All moving objects have angular momentum with respect to some point, but it is generally used to describe the motion of rotating bodies.



Block rotating about point O

2) In statics, we learned that forces not only cause particles to translate but they produce moments (or tendencies to rotate about specified points) in said particles as well . In a similar way, we can **define angular momentum as the moment of the linear momentum of a particle about a specified point.** Angular momentum is a function of the linear momentum of the particle, with the moment arm being drawn from the point of rotation to the particle. For example, in the diagram below, the moment arm would be drawn from the point O to the particle. Mathematically, angular momentum is defined by the cross product of the moment arm and the linear momentum., and it can be evaluated by the 3x3 determinant shown. It is a vector that is perpendicular to the plane defined by the two original vectors. In the example below, the magnitude of $\mathbf{H}_o$ is $(H_o)_z = mvd$ with H_o representing the angular momentum about O.



$$\mathbf{H}_o = \mathbf{r} \times m\mathbf{v} = \begin{vmatrix} i & j & k \\ r_x & r_y & r_z \\ mv_x & mv_y & mv_z \end{vmatrix}$$

Angular momentum of particle about O

3) We can rewrite Newton's second law in the form shown. The resultant force acting on the particle is equal to the time rate of change of the particle's linear momentum.

$$\sum \boldsymbol{F} = \mathrm{m}\,\mathrm{d}\boldsymbol{v}/\mathrm{dt} = \mathrm{d}\mathbf{L}/\mathrm{dt}$$

Similarly, it may be shown that the resultant moment is equal to the time rate of change of the particle's angular momentum.

$$\sum \boldsymbol{M}_{\mathrm{o}} = \mathrm{d}\boldsymbol{H}_{\mathrm{o}}/\mathrm{dt}$$

4) We define angular impulse as the cumulative effect of a moment over time. The particle's velocity, and by extension, angular momentum, vary depending on the magnitude of the moment AND the length of time that the moment is applied to the particle. The relationship between moment application and change of angular momentum is described by the **principle of angular impulse-momentum**. The initial angular momentum plus the sum of all the angular impulses equals the final angular momentum.

$$\sum \int \boldsymbol{M}_{\mathrm{o}}\,\mathrm{dt} = (\boldsymbol{H}_{\mathrm{o}})_2 - (\boldsymbol{H}_{\mathrm{o}})_1$$

or

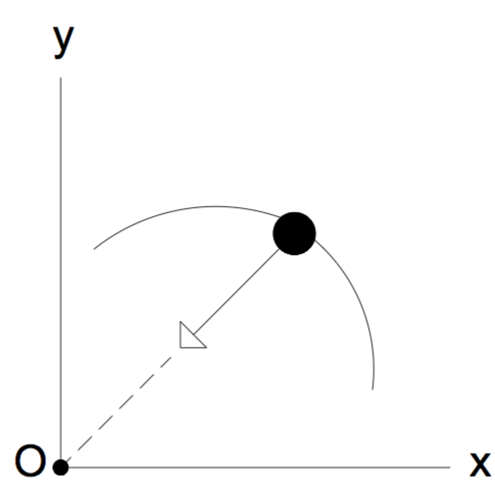
$$(\boldsymbol{H}_{\mathrm{o}})_1 + \sum \int \boldsymbol{M}_{\mathrm{o}}\,\mathrm{dt} = (\boldsymbol{H}_{\mathrm{o}})_2$$

Where the limits of the integral are taken from t_1 to t_2.

5) **Angular momentum** of a particle or system of particles **is conserved when the angular impulse is zero** over a specified interval of time. When the resultant moment about a point is zero, then the angular momentum about that point remains constant. **A central force problem is a problem that lends itself to a conservation of angular momentum analysis,** in that selecting an appropriate point results in a moment of zero for the particle and conservation of angular momentum about that point. Examples of this type of problem include a car rounding a curve with a constant speed, a satellite orbiting the earth, and an object tied to a string being swung around one's head.

$$(\boldsymbol{H}_O)_1 = (\boldsymbol{H}_O)_2$$

In the figure, the central force $\boldsymbol{F}$ is always directed toward point O. Thus, the angular impulse of $\boldsymbol{F}$ about O is always zero, and angular momentum of the particle about O is conserved.



Central force problem

Example 1

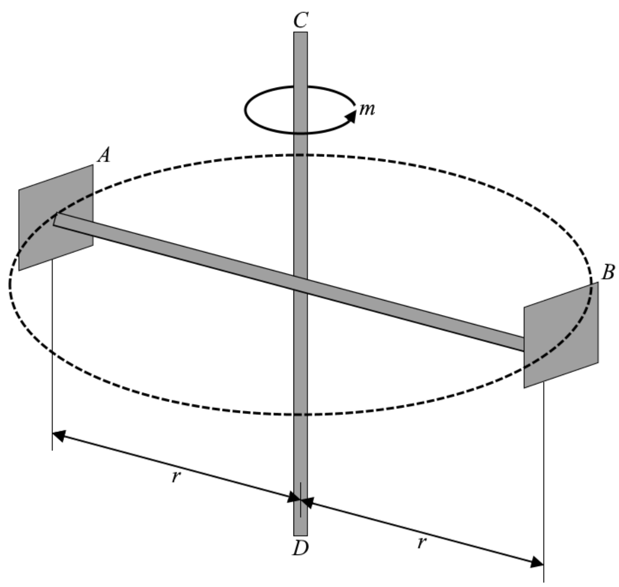
A particle with a mass of 4 kg has a position vector in meters given by $\mathbf{r} = 3t^2\mathbf{i} - 2t\mathbf{j} - 3t\mathbf{k}$., where t is the time in seconds. For t = 3s, determine the magnitude of the angular momentum of the particle about the origin of coordinates.

Example 2

Given: Two 3 kg masses with initial velocities of 5 m/s experience moment of .8 Nm with radius of .4 m

Find: The speed of blocks A and B when t = 3 seconds

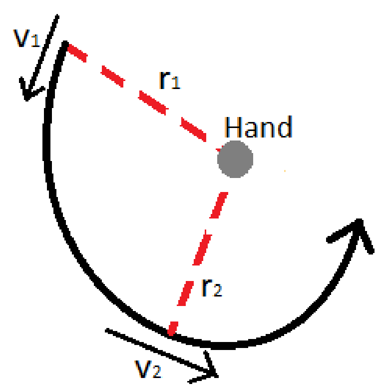
Plan: Apply the Principle of Angular Impulse and Momentum



Group Work

Given: A 2-lb ball is connected to a string which is being swung in a horizontal circular path. When the string has a radius of $r_1 = 8$ ft, the ball has a speed of $v_1 = 15$ ft/s.

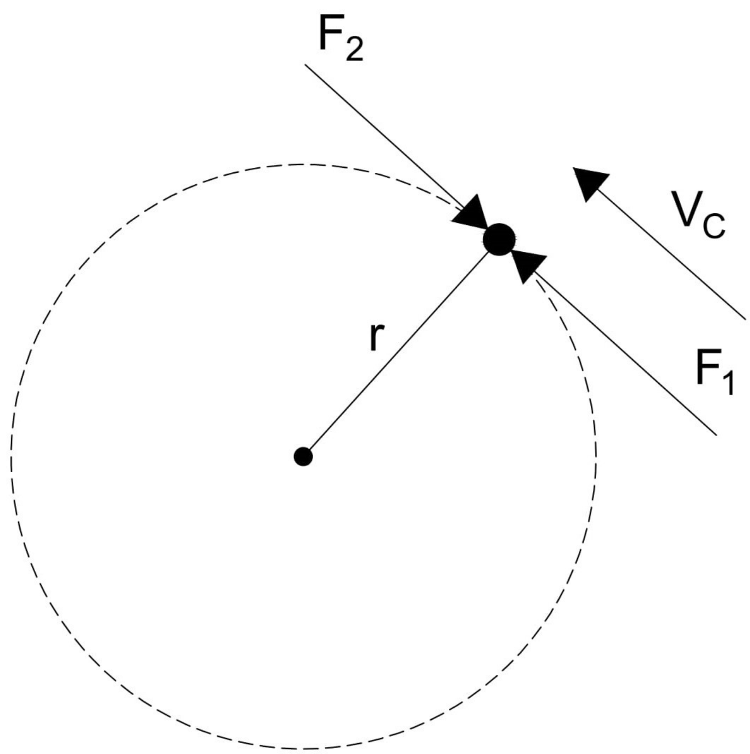
Find: If the string is retracted at a rate of 1 ft/s, find the speed of the ball, v_2, when the string has a radius of $r_2 = 5$ ft and find the work done by the axial force F along the string.



Plan: Apply the equation of conservation of angular momentum to find v_2 at $r_2 = 5$ ft. Then use the principle of work and energy to find the work done by force F.

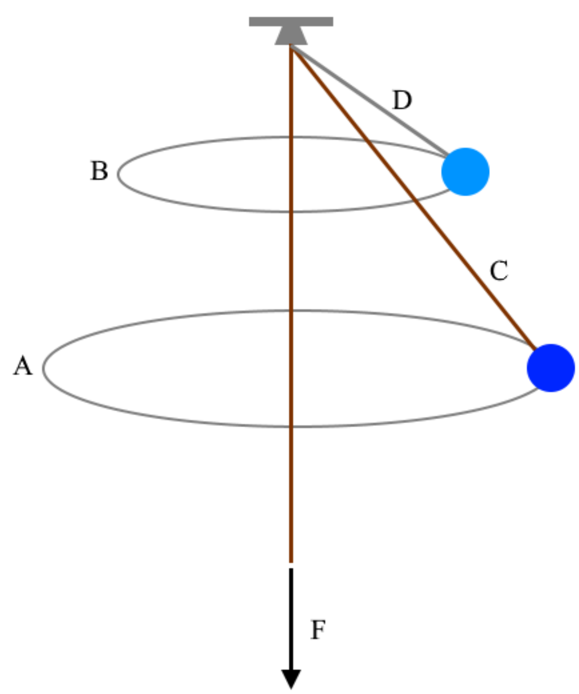
Homework #18

1) A mass of 2 kg is attached to a radial arm with a length of 1.5 m. The mass has a constant velocity V_c = 3 m/s. After 3 seconds has passed, two separate and constant forces of F_1 = 13 N and F_2 = 6 N are applied to the mass perpendicular to the arm. Determine the angular moment and angular velocity 2 seconds after the two constant forces have been applied.



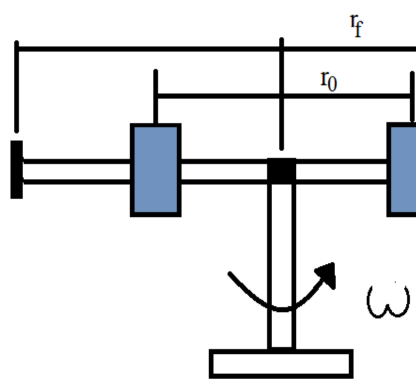
Homework #18

2) A 3-kg ball is attached to the end of a cord. When the ball is given a horizontal velocity of 2.5 m/s, the length of the cord is C = 600 mm, and it begins to move around the horizontal circular path A. If the force P on the cord is increased, the ball rises and is now traveling along the circular path B and the cord shortens to a length D = 300 mm. Determine the speed of the ball around path B and find the work done by the force P.



Homework #18

3) Two blocks with a mass $m=5kg$ are on opposite sides of a rotating rod. At first the blocks are locked in place at a distance of $r_o=2m$ from the rotating axis. The rod initially rotates at an angular velocity of $\omega_o=3rad/s$. Determine the final angular velocity ω_f after the blocks are released and slide down to the end of the rod at a distance of $r_f=4m$ from the rotating axis.

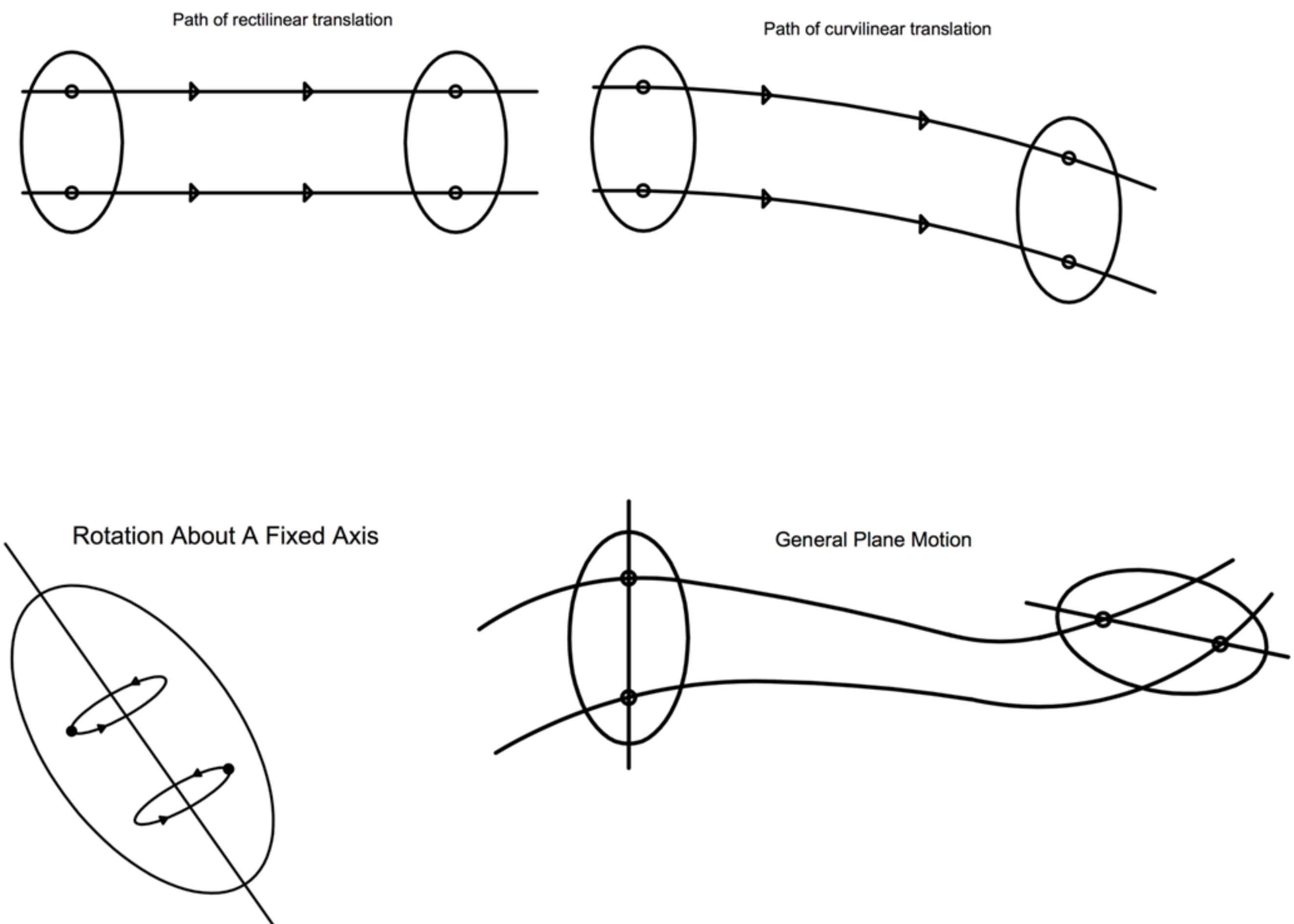


Rigid Body Motion

Lesson 19

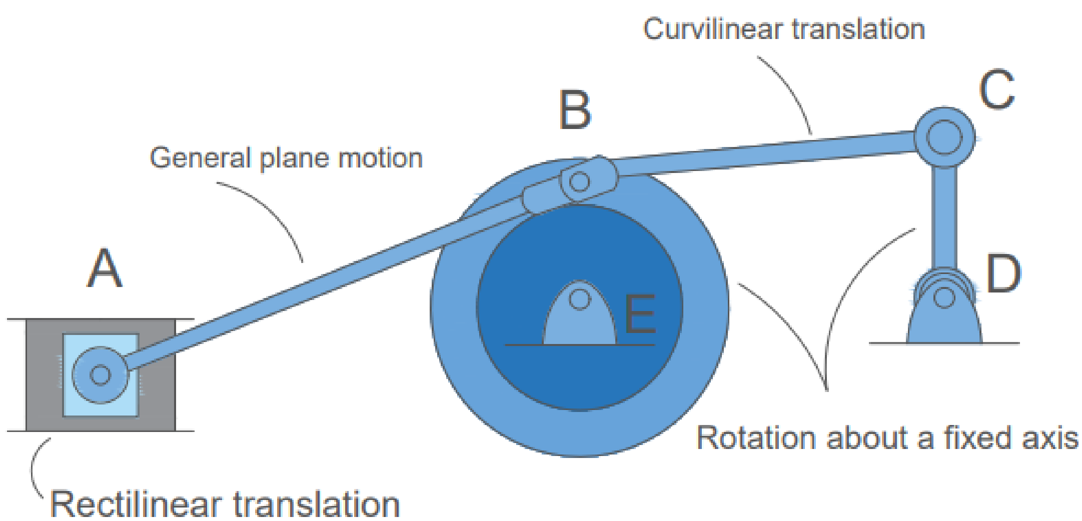
1) We begin to analyze the motion of rigid bodies. Rigid bodies have size and shape as well as mass. **The rotation of a rigid body must be considered along with translation**. We need to consider angular motion as well as linear motion.

2) A rigid body can move in one of three ways; **pure translation (either rectilinear or curvilinear), pure rotation, or general plane motion.**



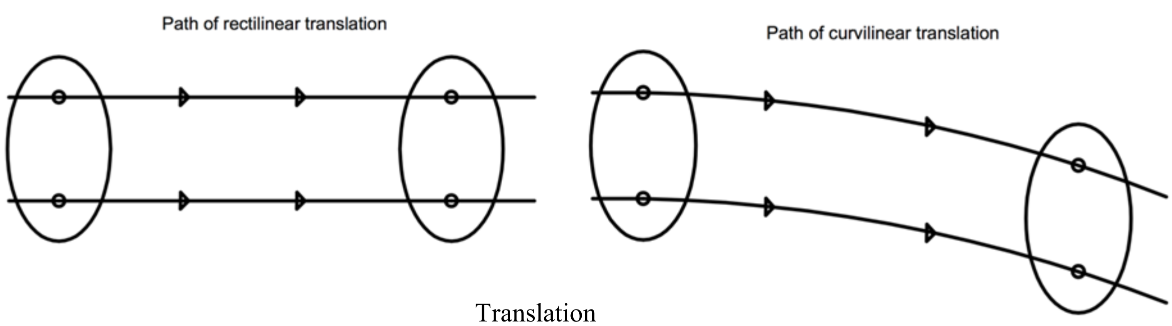
Three types of rigid body motion

3) Here is a mechanism comprised of several parts that are exhibiting all three types of rigid body motion. The piston and the right connecting rod are experiencing **translation**. The wheel and the crank are experiencing **pure rotation**, and the left connecting rod is experiencing **general plane motion**.



Mechanism illustrating three types of motion

4) Pure translation occurs when a body goes from one location to another without changing its orientation. The paths taken by each of the points are parallel to one another. Every line segment on the body remains parallel to its original direction during the motion. When all points move along straight lines, the motion is called **rectilinear translation**. When the paths of motion are curved lines, the motion is called **curvilinear translation.**



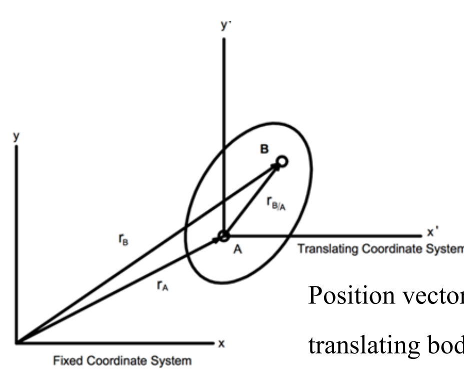
Translation

The kinematics of rigid bodies in pure translation is easy. **All points in the rigid body move with the same velocity and acceleration.** This is because all points move in paths that are parallel to one another. This is derived by expressing the absolute position of one point as the absolute position of another point plus the relative position of the second point with respect to the first point. Differentiating this relationship results in both points having the same velocity and acceleration.

Position: $\boldsymbol{r}_B = \boldsymbol{r}_A + \boldsymbol{r}_{B/A}$

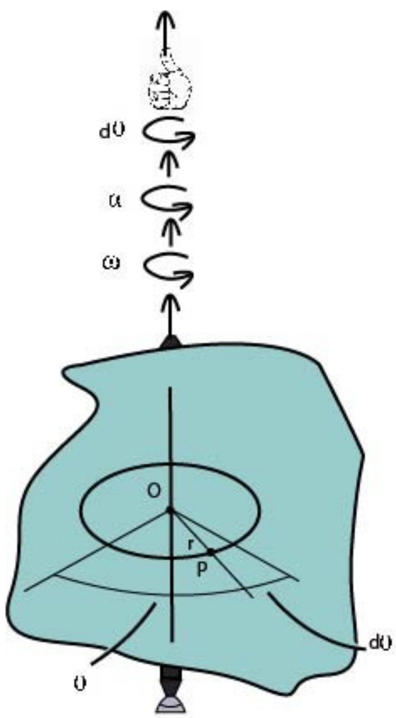
Velocity: $\boldsymbol{v}_B = \boldsymbol{v}_A$

Acceleration: $\mathbf{a}_\mathrm{B} = \mathbf{a}_\mathrm{A}$



Position vectors of translating body

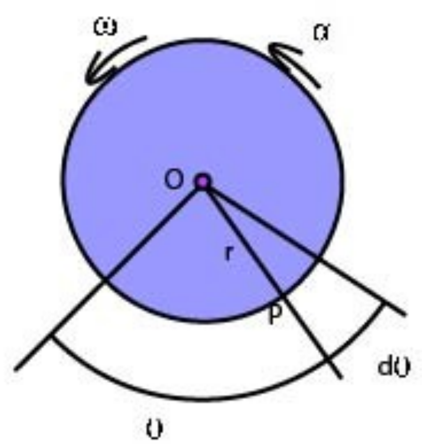
5) In pure rotation, **each point in the body moves in a circular path about the axis of rotation**. In describing the kinematics of the body as a whole, we define angular position, angular velocity and angular acceleration. These are related to one another in the same way as s, v and a are related in rectilinear motion.



1 revolution = (2π) radians

$\omega = d\theta/dt$ *(rad/s)*

$\alpha = d^2\theta/dt^2 = d\omega/dt$ *or* $\alpha = \omega(d\omega/d\theta)$



Elevation Fixed Axis Rotation Plan

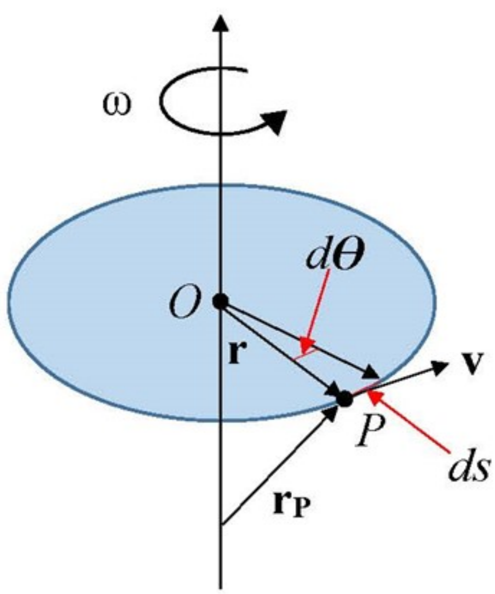
If the angular acceleration is constant, we can show that the three constant acceleration relationships developed for rectilinear motion can be applied to pure rotation problems, simply substituting θ, ω, and α for s, v, and a.

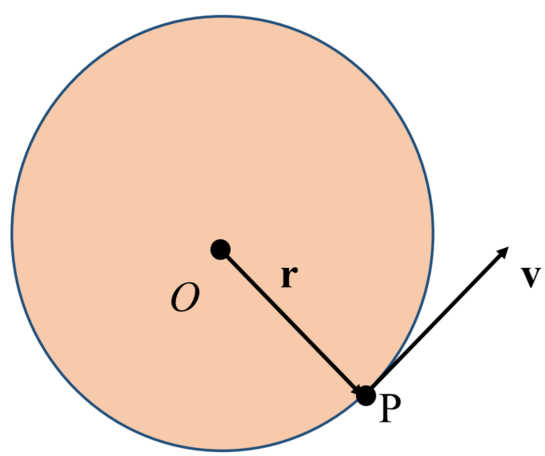
$$\omega = \omega_0 + \alpha_C t$$

$$\theta = \theta_0 + \omega_0 t + 1/2\, \alpha_C t^2$$

$$\omega^2 = (\omega_0)^2 + 2\alpha_C (\theta - \theta_0)$$

In addition to describing the motion of the rigid body as a whole, we are interested in describing the motion (velocity and acceleration) of individual points located on the rotating rigid body. We take advantage of what we know about the nature of particles moving in circular paths. If a body rotates an amount $d\theta$, point P will displace an amount $ds = rd\theta$. Knowing that $v = ds/dt = rd\theta/dt = \omega r$. **Therefore the velocity of any point will be tangent to its circular path, of magnitude ωr.** The vector formulation is $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$.





Velocity of any point tangent to its circular path

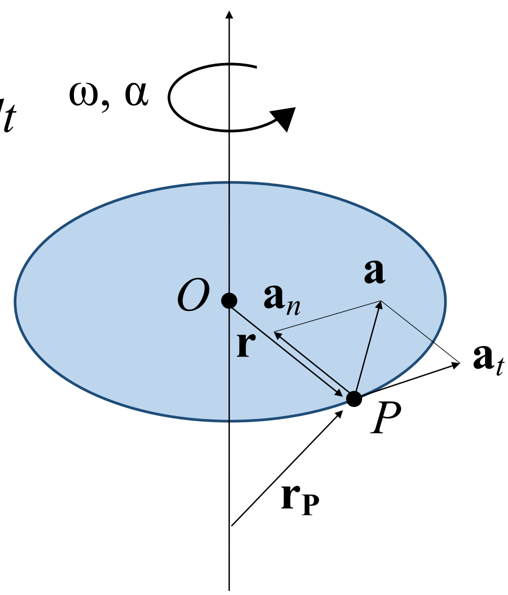
The acceleration of any point moving in a circular path is most easily expressed in normal-tangential coordinates. The magnitude of the tangential acceleration component is $a_t = dv/dt = d\omega r/dt = \boldsymbol{\alpha r}$. The magnitude of the normal acceleration component is $a_n = v^2/r = \omega^2 r^2/r = \boldsymbol{\omega^2 r}$.

$$\boldsymbol{a} = d\boldsymbol{v}/dt = d\boldsymbol{\omega}/dt \times \boldsymbol{r}_P + \boldsymbol{\omega} \times d\boldsymbol{r}_P/dt$$

$$= \boldsymbol{\alpha} \times \boldsymbol{r}_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_P)$$

$$\mathbf{a} = \boldsymbol{\alpha} \times \boldsymbol{r} - \omega^2 \boldsymbol{r} = \boldsymbol{a}_t + \boldsymbol{a}_n$$

$$a = [\,(a_t)^2 + (a_n)^2\,]^{1/2}$$



Acceleration of any point

In vector form, $\mathbf{a} = \boldsymbol{a} \times \boldsymbol{r} - \boldsymbol{\omega}^2 \boldsymbol{r}$.

Procedure for Rotation about a Fixed Axis

1) Establish a sign convention along the axis of rotation.

2) If a relationship is known between any two of the variables (α, ω, *θ, or t), the other variables can be determined from the equations:* $\omega = d\theta/dt$ $\quad \alpha = d\omega/dt$ $\quad \alpha\, d\theta = \omega\, d\omega$

3) If α is constant, use the equations for constant angular acceleration.

4) To determine the motion of a point, the scalar equations $v = \omega\, r$, $a_t = \alpha\, r$, $a_n = \omega^2 r$, and $a = [\, (a_t)^2 + (a_n)^2 \,]^{1/2}$ can be used.

5) Alternatively, the vector form of the equations can be used (with ***i**, **j**, **k** components).*

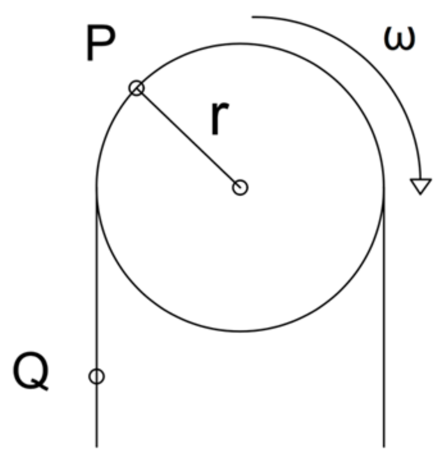
$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}_P = \boldsymbol{\omega} \times \boldsymbol{r}$$

$$\mathbf{a} = \mathbf{a}_t + \mathbf{a}_n = \boldsymbol{\alpha} \times \boldsymbol{r}_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_P) = \boldsymbol{\alpha} \times \boldsymbol{r} - \omega^2 \boldsymbol{r}$$

Example 1

Given: Starting from rest, a rope wrapped around a pulley is pulled downward. The clockwise angular velocity of the pulley increases at a constant rate to 10 rad/s after a period of 4 seconds. r = 0.3 m, ω_0 = 0 rad/s, ω = 10 rad/s at 4 seconds

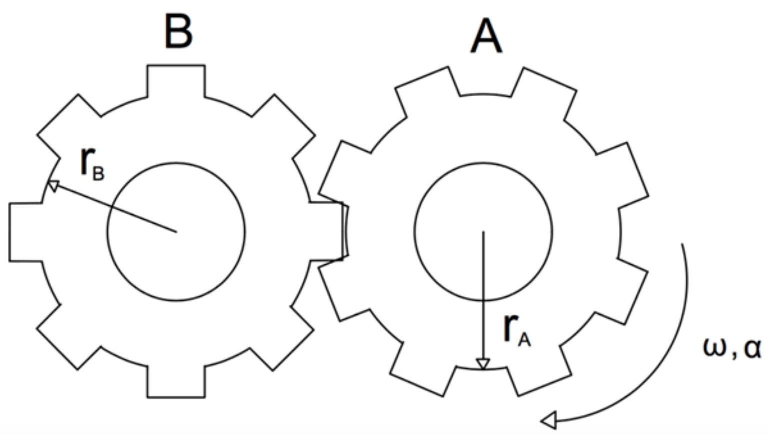
Find: The velocities and accelerations at points P and Q after 7 seconds.



Lesson 19 Group Work

Given: Gear A starts from an angular velocity of 1 rad/second, has a constant angular acceleration of 2 rad/s^{2,} and a radius of 1 meter. Gear B has a radius of 2 meters.

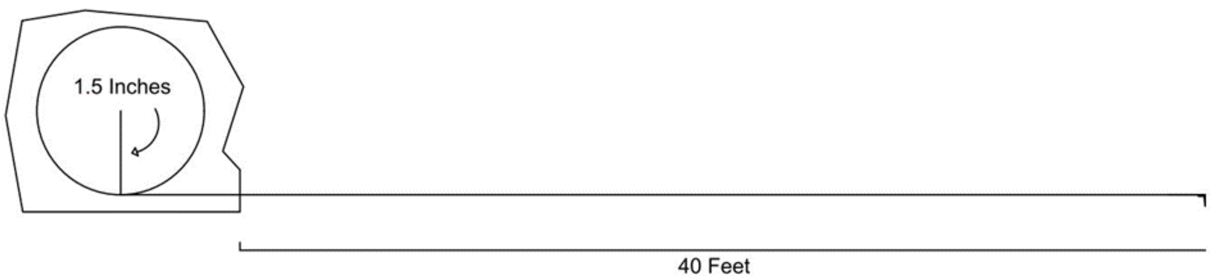
Find: The angular acceleration of gear B after five seconds has passed.



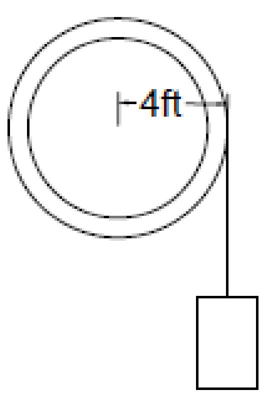
Plan: Find the final angular velocity of gear A then the final angular velocity of gear B using constant angular acceleration kinematics and the knowledge that the linear velocities of the gears will equal one another so $\omega_A * r_A = \omega_B * r_B$. Find initial angular velocity of gear B and then use constant angular acceleration kinematics to solve for the angular acceleration of gear B.

Homework Assignment #19

1) Two builders are measuring a hallway with a retractable tape measure. Having completed the 40-foot measurement, the builder holding the end of the tape measure lets go, and it comes whizzing back into the casing with an acceleration of 18 ft/s^2. If the radius of the tape measure inside the casing is 1.5 inches, what are the angular acceleration and angular velocity of the tape at the instant before it has completely retracted? What is the linear velocity of the tape at that time?



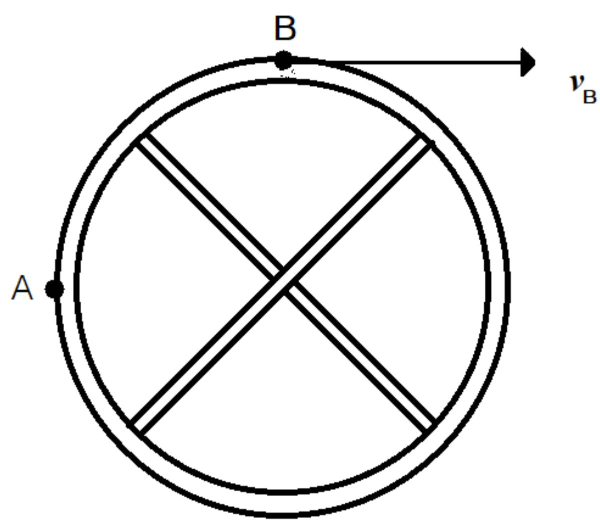
2) A weight is attached to a cord which is wound around a wheel. If it moves with an initial velocity of 2 ft/s and an acceleration of 7 ft/s^2, determine the angular acceleration of the drum and its angular velocity after 10 seconds.



Lesson 19[2] Group Work

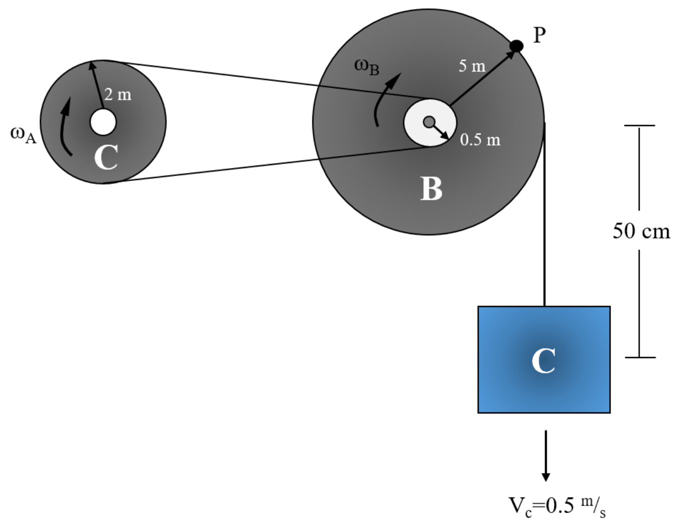
Given: A wheel with radius 1 foot is at rest. A force is applied to it and after completing 1 revolution point B has a a velocity of 8 ft/s at the instant shown.

Find: What is the instantaneous acceleration of point A? (Give your solution in vector form).

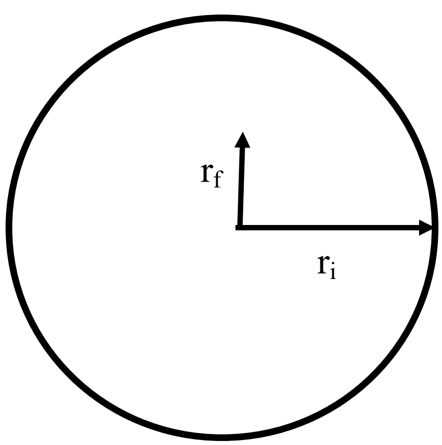


Homework Assignment 19[2]

1) Block C is attached to a rope wrapped around belt B. The block starts from rest and after falling 50 cm has a velocity of 0.5 m/s. The rope unravels with no slip and causes belt B and C to rotate. Block C falls with a constant acceleration. For the instant shown, find the angular velocity of belt B and A.



2) A record is spinning at v =1.3 m/s where the needle is located. Find the angular speed of the record when the first song plays at r_i=250mm, and when the final song ends at r_f=100mm. If the length of the album is t=35 minutes, and α is constant, how many revolutions does the record make?

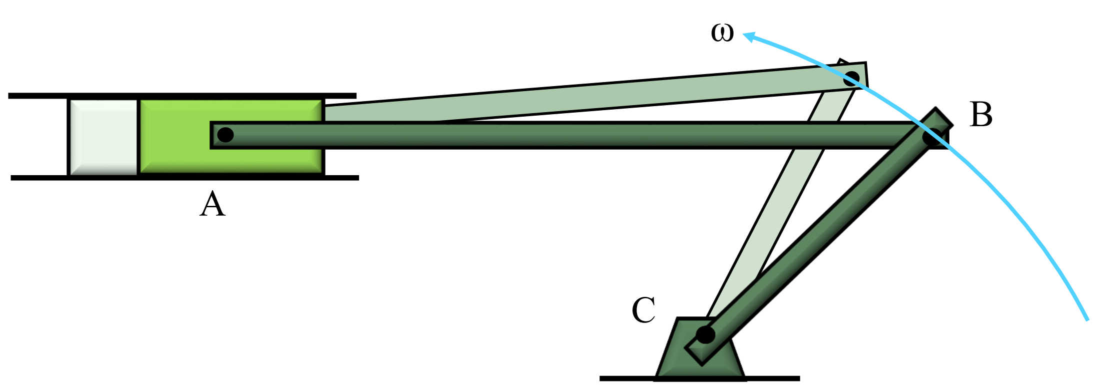


General Plane Motion

Lesson 20

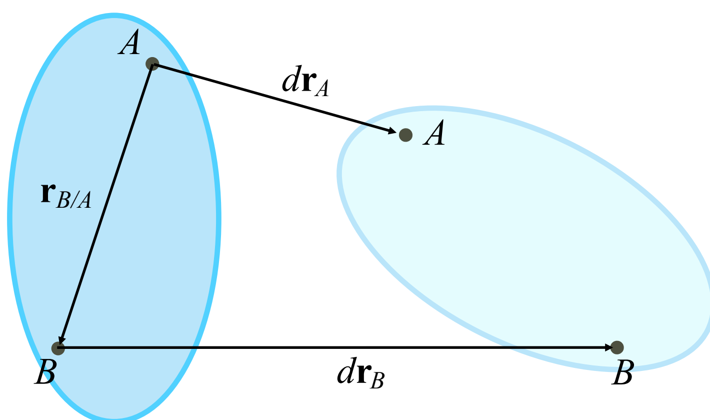
1) In general plane motion, a rigid body can both translate and rotate. In the previous lesson, we looked at the cases of pure translation and pure rotation. Let's now look at the case where we can have both, we refer to that as **general plane motion**.

2) To illustrate the difference, we see a mechanism that consists of a slider block (A), a connecting rod (BA), and a link (BC). The slider block experiences pure translation in the horizontal direction, the link experiences pure rotation around point C, and the connecting rod experiences general plane motion, both translation and rotation. We would like to be able to find and relate the velocities and A and B, and the angular velocity of the general plane member.

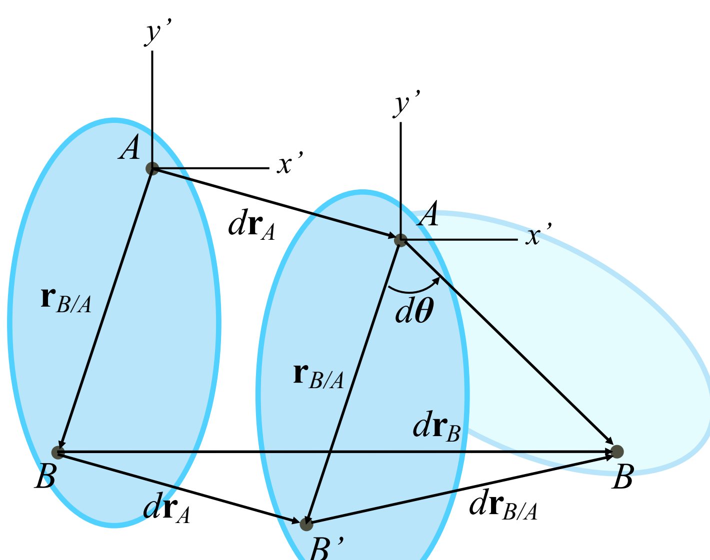


Member AB experiencing general plane motion

3) To analyze the kinematics of general plane motion, we separate the motion into a **translation component** and a **rotation component**. Observe in the diagram below how points A and B have different displacements. Choose a point whose motion we know something about, such as point A. First, translate the whole body by the amount the A displaces, labeled $d\boldsymbol{r}_A$. Notice how point B’ translates the same amount. Second, rotate the whole body around point A. This is labeled $d\boldsymbol{\theta}$, and B’ rotates along a circular path until it reaches its final position B. Observe from the diagram how $d\boldsymbol{r}_B = d\boldsymbol{r}_A + d\boldsymbol{r}_{B/A}$.

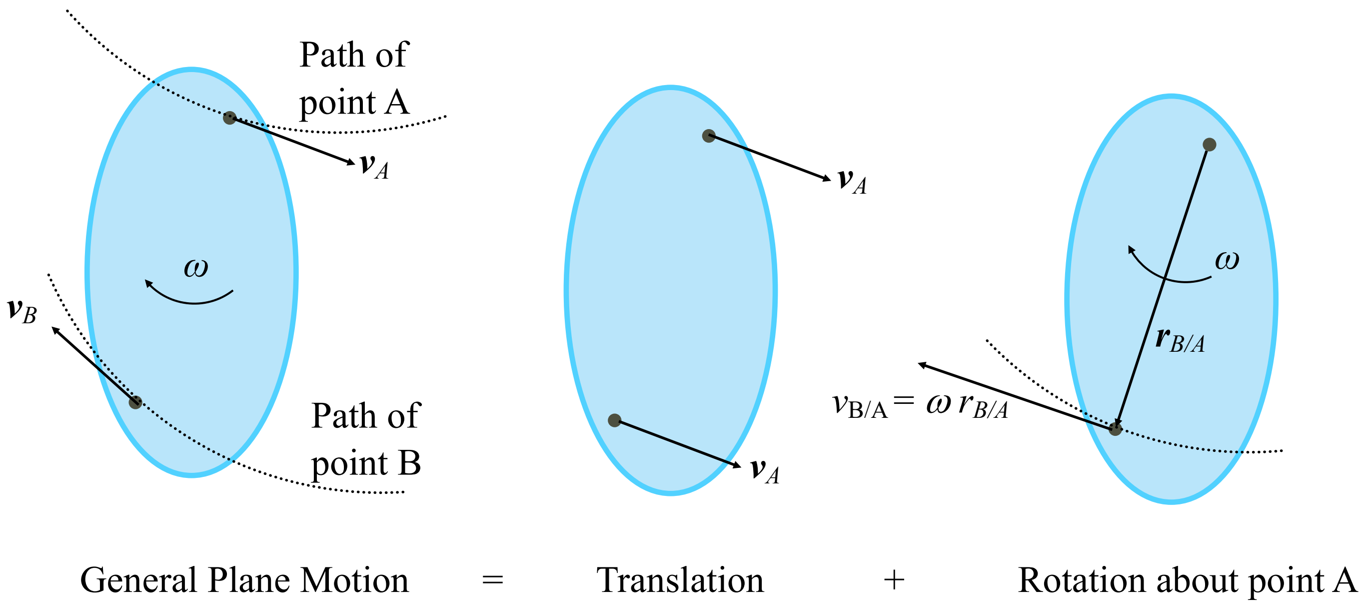


General plane motion



Translation Rotation

4) By differentiating the relative displacement equation, we can relate the velocities of points A and B. If the velocity at point A is $\boldsymbol{v}_A$, then during pure translation, the velocity at point B is also $\boldsymbol{v}_A$. During the fixed axis rotation about the base point A, the velocity at point B is $\omega \boldsymbol{r}_{B/A}$. By summing these components, we obtain the equation for the relative velocity between two points on a rigid body in general plane motion: $\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{\omega} \times \boldsymbol{r}_{B/A}$



The velocity at B is given as:

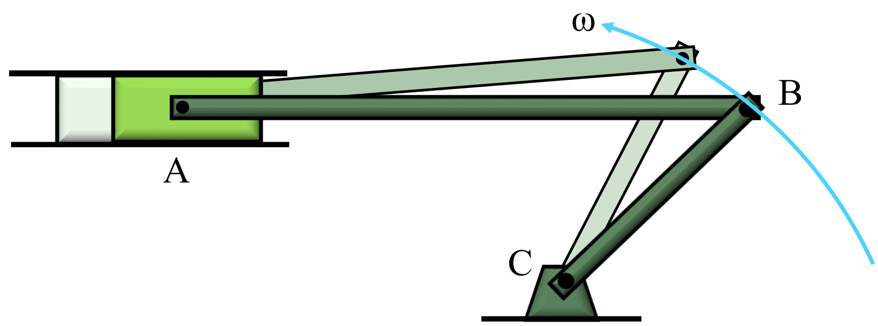
$$(d\boldsymbol{r}_B/dt) = (d\boldsymbol{r}_A/dt) + (d\boldsymbol{r}_{B/A}/dt) \text{ or } \boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{v}_{B/A}$$

Since the rigid body is purely rotating about A,

$$\mathbf{v}_{B/A} = d\boldsymbol{r}_{B/A}/dt = \boldsymbol{\omega} \times \boldsymbol{r}_{B/A}$$

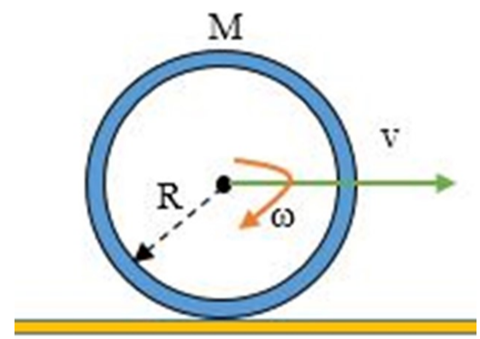
Here $\boldsymbol{\omega}$ will only have a $\boldsymbol{k}$ component since the axis of rotation is perpendicular to the plane of translation.

5) This relative velocity equation relates the velocities at two points, as well as the angular velocity of a member in general plane motion. In the mechanism with pinned members AB and BC below, we know that the direction of $\mathbf{v}_A$ is horizontal, and that the direction of $\mathbf{v}_B$ is tangent to the circular path rotating around point C. $\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{\omega} \times \boldsymbol{r}_{B/A}$



Application of relative velocity equation to AB

6) A wheel is another classic example of general plane motion, experiencing both translation and rotation. The velocity at the center of a wheel must be horizontal, as the center always is the same radial distance from the ground. If there is no slipping between the ground and the point on the wheel in contact with it, then the instantaneous velocity at that point is zero, as the ground does not move. Now we have information on the velocity of two of the points on the wheel, so we can use the relative velocity equation to find the velocity at the center and the angular velocity of the wheel. $\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{\omega} \times \boldsymbol{r}_{B/A}$



Rolling problem example

7) These are the steps to follow in using the relative velocity equation for general plane motion.

Vector Analysis Procedure:

1) Establish the fixed x - y coordinate directions and draw the kinematic diagram of the body, showing the vectors $\boldsymbol{v}_A$, $\boldsymbol{v}_B$, $\boldsymbol{r}_{B/A}$ and $\boldsymbol{\omega}$. If the magnitudes are unknown, the sense of direction may be assumed.

2) Express the vectors in Cartesian vector form (CVN) and substitute them into $\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{\omega} \times \boldsymbol{r}_{B/A}$. Evaluate the cross product and equate respective $\boldsymbol{i}$ and $\boldsymbol{j}$ components to obtain two scalar equations.

3) If the solution yields a negative answer, the sense of direction of the vector is opposite to that assumed.

Example 1

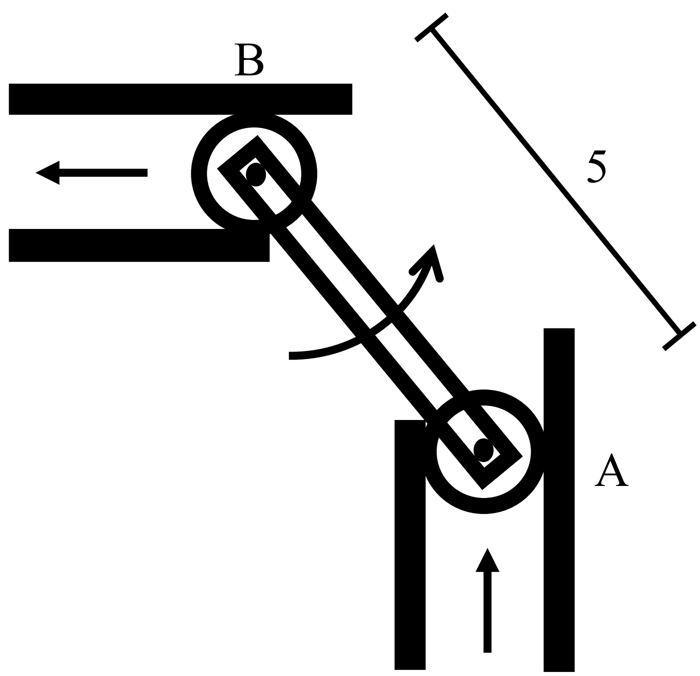
A metal arm with two rotational balls attached to each end of the arm. One ball is vertical and the other is horizontal, the arm is turning counterclockwise sliding ball A up at $v_a = 6$ m/s and pushing ball B to the left. Find velocity of ball B at an instant of $\theta = 45$ degrees.

Given: The arm is at 45 degrees respect to both balls turning counterclockwise. $l_{AB} = 5$ m

Find: Velocity of ball B at the instant $\theta = 45$ degrees.

Plan: Set the velocity and distance of the arm between each balls in vector form.

Use $\mathbf{v}_b = \mathbf{v}_a + \boldsymbol{\omega} \times \mathbf{r}_{b/a}$ to find $\mathbf{v}_b$.

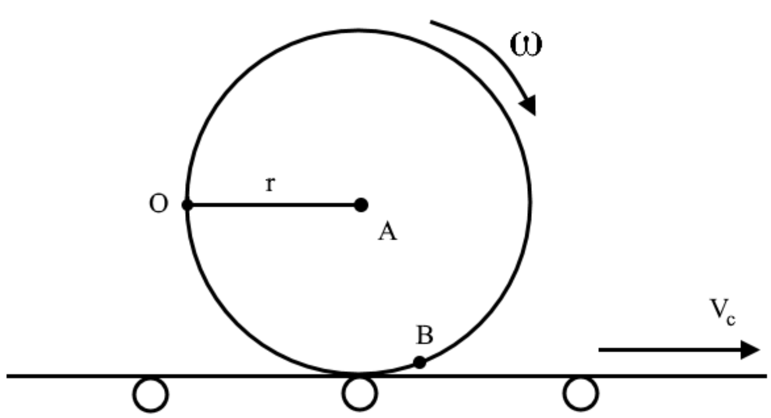


Example 2

Given: The sphere shown in the figure is rolling without slipping on a conveyor belt which is moving at 4 ft/s. The radius of the sphere is 1 ft.

Find: The velocity of point O if the sphere has a clockwise angular velocity of 17 rad/s at the instant shown.

Plan: Since there is no slipping, point B will have the same velocity as the conveyor belt. Apply the relative velocity equation to B & O to determine V_O.

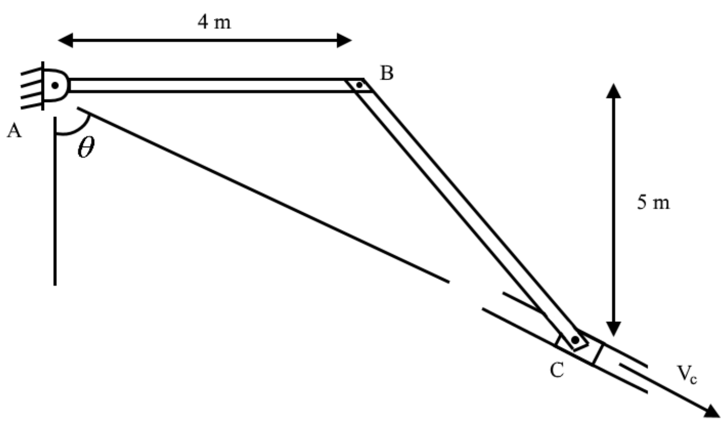


Lesson 20 Group Work

Given: Slider block C moves at a velocity of 6 m/s down the incline.

Find: Determine the angular velocity of links AB and BC at the instant shown if $\Theta = 60^{\circ}$.

Plan:



1. Find V_B in terms of ω_{AB}

2. Apply the relative velocity equation

3. Equate i components to solve for ω_{AB}

Lesson 20 Group Work [2]

When link AB rotates, gear F oscillates at a certain velocity. If AB has an angular velocity of ω_{AB} = 10 rad/s, calculate the angular velocity of gear F. Assume gear E is part of the arm of CD and pinned to D at the fixed point.

Given: $r_{B/A}$ = 0.08m, $r_{D/C}$ = 0.25m, $r_{C/B}$ = 0.15m, r_F = 0.05m, r_E = 0.2m, ω_{AB} = 10 rad/s

Find: ω_F

Plan:

Calculate the velocity at C:

Write the position vector $\mathbf{r}_{C/B}$:

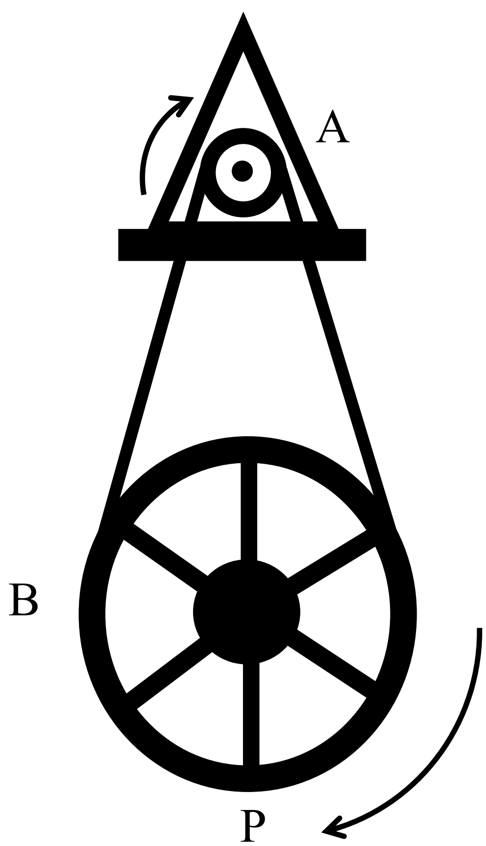
Write the velocity equation at point C:

Equate the **i** and **j** coefficients:

Solve for ω_F using the equations for the gear:

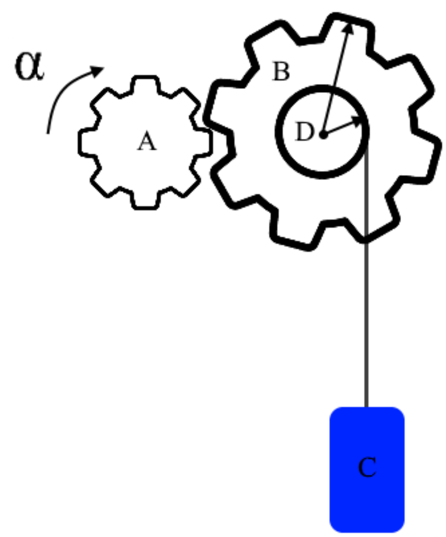
Homework Assignment 20

1. The motor turns the wheel with the attached cable. Pulley A rotates from rest with a constant angular acceleration of $\alpha_A = 3$ rad/s^2. Determine the magnitudes of the velocity and acceleration of point P on the wheel after the pulley has completed 4 revolutions.



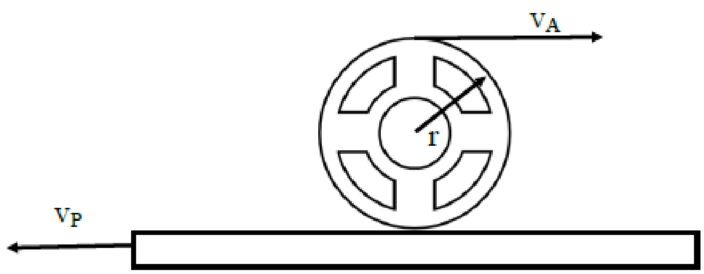
Homework Assignment 20

2. A constant angular acceleration of $\alpha = 6\ \text{rad/s}^2$ is acting on gear A. Determine the velocity of the cylinder C after 3 seconds. The cord that is connected to the cylinder is wrapped around gear D and then gear B. Gear A, Gear B and Gear D have radii of 50 mm, 200 mm and 100 mm respectively.

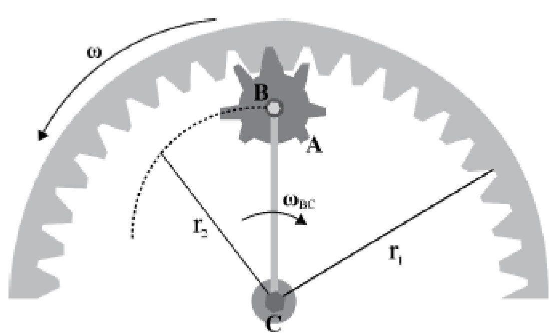


Homework Assignment 20 [2]

1. A rope wraps around a wooden spool and unwraps at the velocity v_A=5m/s while the platform moves in the opposite direction at 3m/s. Find the angular velocity ω of the spool given r=0.4m.



2. Gear A is pinned at B. Link BC rotates clockwise with an angular velocity of 12 rad/s as the outer gear rotates counterclockwise with an angular velocity of 1 rad/s. Find the angular velocity of Gear A.



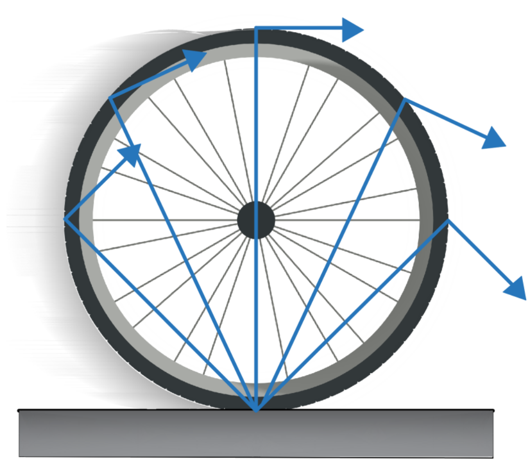
Instantaneous Center of Zero Velocity

Lesson 21

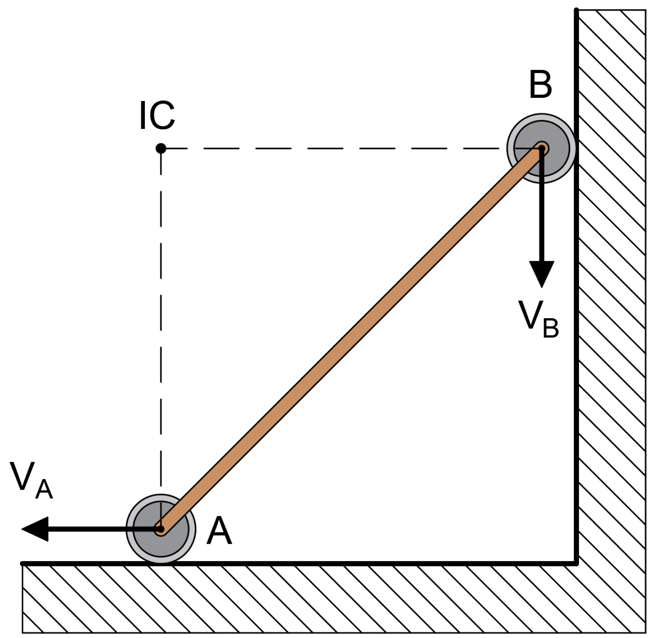
1. Now that we have used the relative velocity equation to relate angular velocity to the velocities of two points, we will conduct the same analysis through an **alternative method.**

2. First, locate the unique point in the system that momentarily has zero velocity. Then, the body can be considered to be in pure rotation about an axis through the point of zero velocity. This method, know as the **instantaneous center of zero velocity,** makes it much easier to find velocities at different points on the body.

3. When considering a wheel rolling without slipping, the instantaneous center of zero velocity is at the point of contact with the ground. The wheel appears to rotate about this point at that instant. This illustrates the principle that **the rigid body appears to rotate about this instantaneous center** (IC). This means that we can treat these situations as pure rotation problems and solve for velocities the same way we would for pure rotation.



Wheel rolling without slipping



Determination of IC of member AB

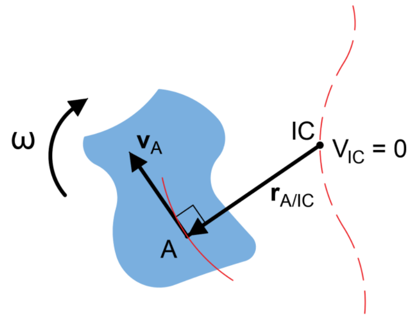
4. For other cases, the instantaneous center must be located using our knowledge of the velocities of a body in pure rotation. The velocity of each point in a body experiencing pure rotation is perpendicular to the position vector from the axis of rotation. The instantaneous center can be located by simply taking a perpendicular line from each velocity vector and locating the point where these lines intersect.

5. For every general plane motion rigid body, it can be shown that there is an instantaneous center of zero velocity.. If this point can be located, the situation can be treated as pure rotation, making it much easier to solve for any unknown velocities of other points on the body.

6. The first step in this method is to determine **the location of this instantaneous center**. There are three general cases to consider.

3 Cases to Consider

I. The angular velocity (ω) of the body and the velocity (v_A) of a point on the body are given.

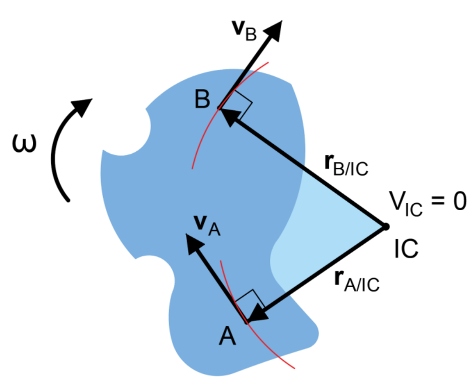


Location of IC
knowing $\mathbf{v}_A$ and ω

Because the body is rotating about the IC, it must be on a line perpendicular to v_A. Since $v_A = \omega r$, and we are given everything but r, the distance can be solved for as v_A/ω.

II. The directions of the velocities of two points on the body are given.

The location of the IC is at the intersection of the two lines perpendicular to the velocities' lines of action.

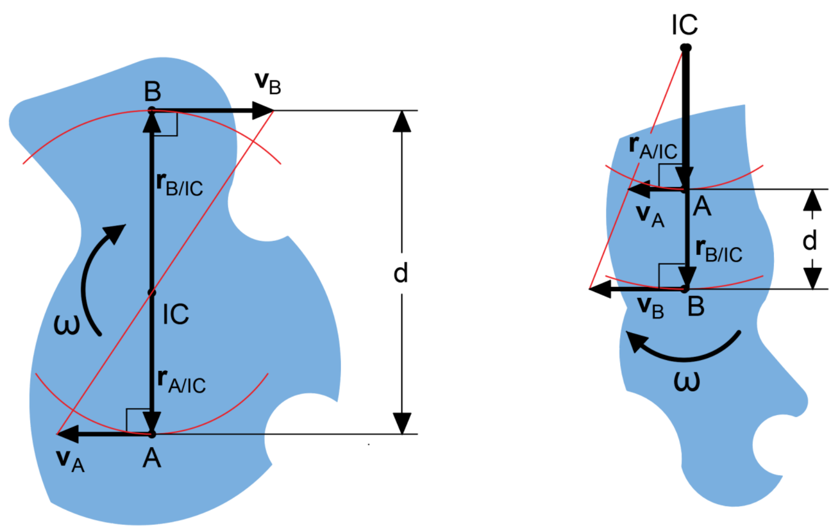


Location of IC
knowing the direction
of $\mathbf{v}_A$ and $\mathbf{v}_B$

3 Cases to Consider (cont.)

III. The magnitude and direction of <u>two parallel velocities</u> from two points on the body are given.

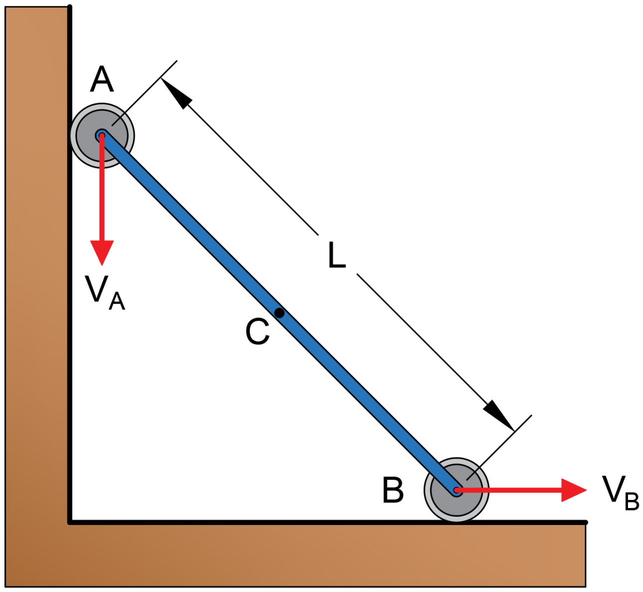
Because $v = \omega r$, the position vectors from the IC to each of the points needs to be proportional to the magnitudes of the velocity vectors. This results in **similar triangles**. By connecting the tips of the velocity vectors and finding where that line intersects with the perpendicular line from the velocities, we can locate the IC.



Location of IC
knowing $\mathbf{v}_A$ and $\mathbf{v}_B$

7. After the IC is located, the problem defaults to fixed axis rotation. The velocity of any other point on the body can be found using $\boldsymbol{v} = \boldsymbol{\omega r}$, with the **direction** of the velocity being **perpendicular to the position vector** from the IC to the point of interest.

Example



Given: A rod is rolling down a wall and across the ground with an angular velocity of 0.5 rad/s. The rod has a length L of 8 m and a center C. Point B has already moved 5 m to the right.

Find: The velocities at points A, B, and C.

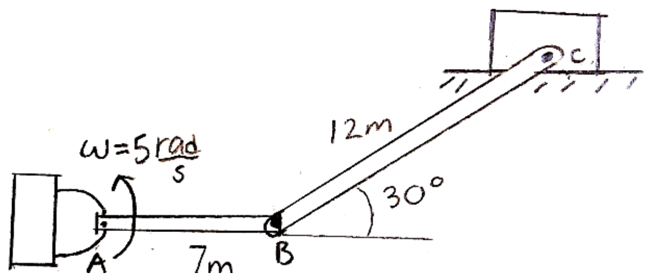
Plan:

1) Locate the instantaneous center of zero velocity.

2) Find the distances from the IC to each point.

Group Problem # 21

Given: Bar AB rotates at ω_{AB}= 5 rad/s determine the angular velocity of BC and determine the velocity at C.



Find: The velocity of point B and C

Plan:

Draw diagram of bar AB and locate IC:

Determine v_b:

Draw diagram of bar BC:

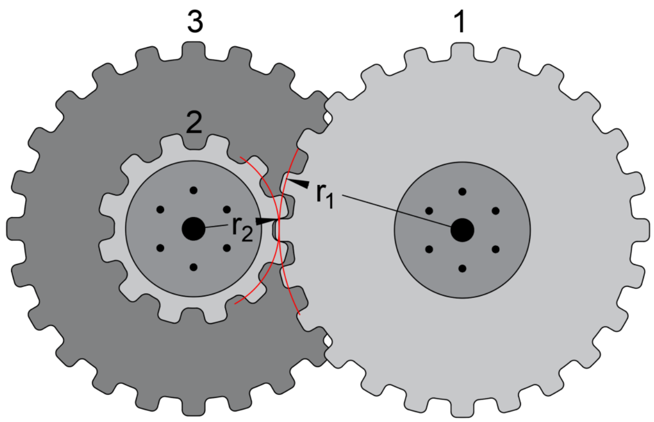
Determine ω_{BC}:

Determine v_C:

Group Problem # 21[2]

Gear 1 is engaged with Gear 2 which is rigidly attached to Gear 3. The angular velocity of Gear 1 is 150 rpm and the radius of Gears 1 and 2 are 20mm and 5mm respectively. What is the angular velocity of Gear 2 in rad/s?

Plan:



Convert the angular velocity of Gear 1 to rad/s:

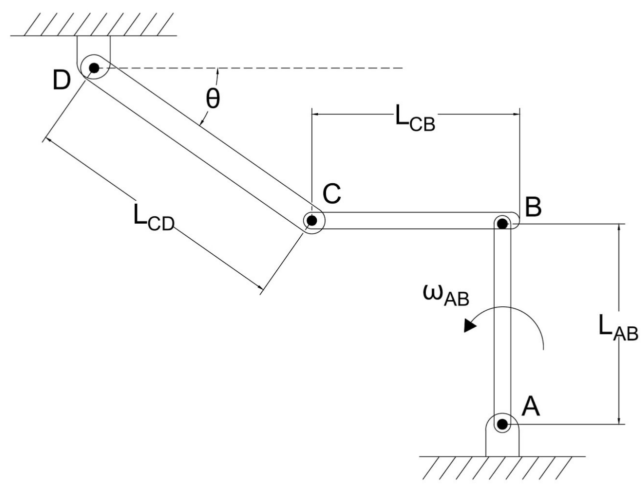
Calculate the linear velocity of Gear 1:

Determine the linear velocity of Gear 2:

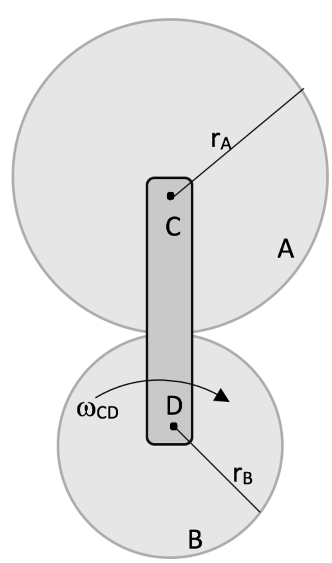
Calculate the angular velocity of Gear 2:

Homework Assignment # 21

1. In the diagram, $L_{AB} = 0.5$ m, $L_{CB} = 0.4$ m, $L_{CD} = 1.5$ m, and $\theta = 30°$. If rod AB rotates with an angular velocity of $\omega_{AB} = 4$ rad/s, find the angular velocities of rods CB and CD.

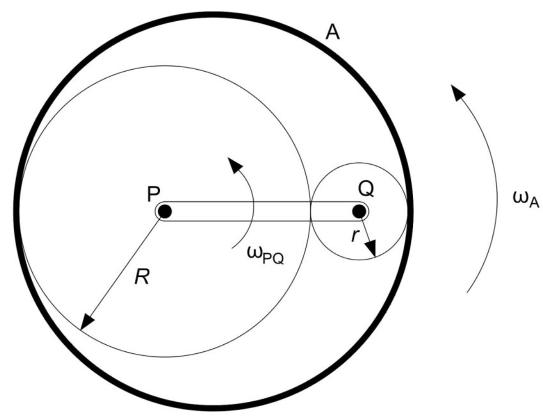


2. Cylinder A rolls on the fixed cylinder B without slipping. If bar CD rotates with an angular velocity of $\omega_{CD} = 5$ rad/s, determine the angular velocity of A and the angular velocity of B.



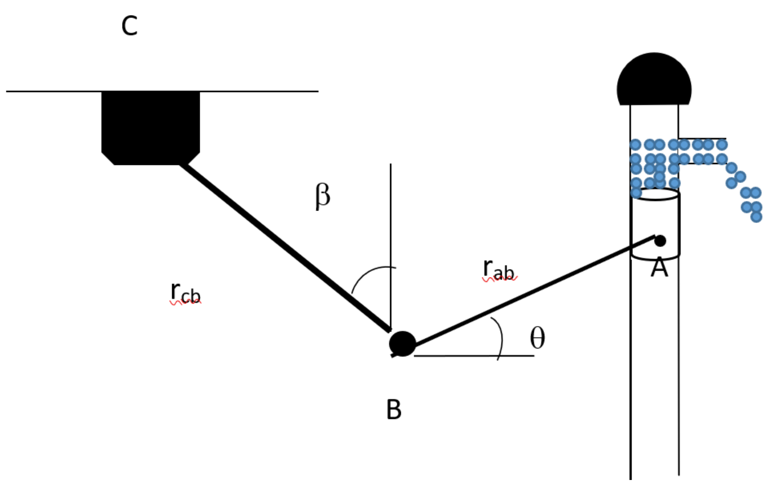
Homework Assignment # 21[2]

1. Two discs are are surrounded by a ring A. The fixed disc P of radius R = 125 mm and the disc Q of radius r = 50 mm are connected by an arm at their axes. The angular velocity ω_{PQ} of the arm is 5 rad/s. Determine the angular velocity of ring A surrounding the two inner discs. Assume there is no slipping between the rings or discs.



Homework Assignment # 21[2]

2. The water pump is pumping water out of the well with the mechanism given. The angles of the bars are θ=38° and ß=40°. The length of bar r_{ab}= 0.8m and bar r_{cb}= 1.2m. Determine the angular velocity of the link r_{bc} at the instant shown if the piston A is moving upward at 6 m/s.

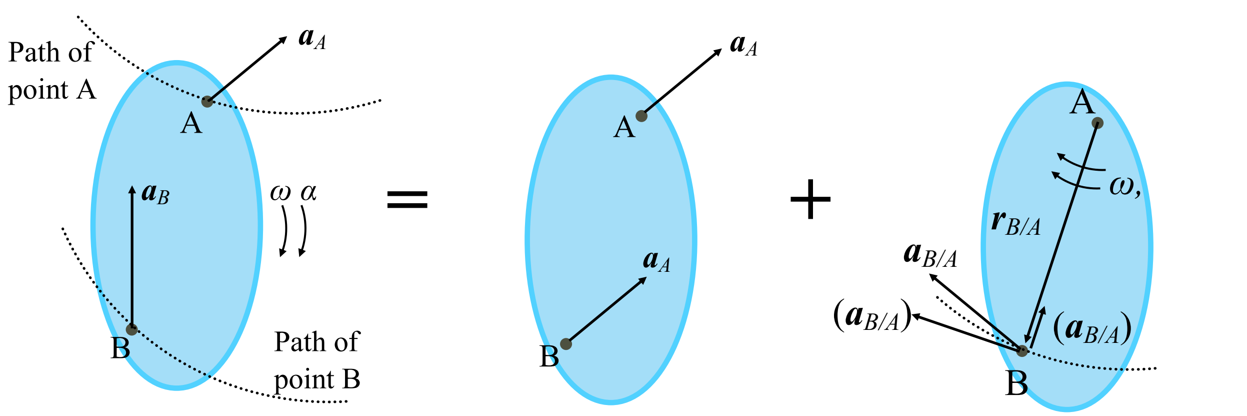


Relative Motion Analysis-Acceleration

Lesson 22

1. We have analyzed the kinematics of rigid bodies undergoing general plane motion with respect to velocity. To complete our study of general plane motion kinematics, we need to understand and describe the body's **acceleration**.

2. Recall that when we analyzed general plane motion, we separated the body's motion into **translational** and **rotational** components. We then related the velocities between two points on the body and derived the equation $\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{v}_{B/A}$. We will use a similar approach to analyze the acceleration of two points.

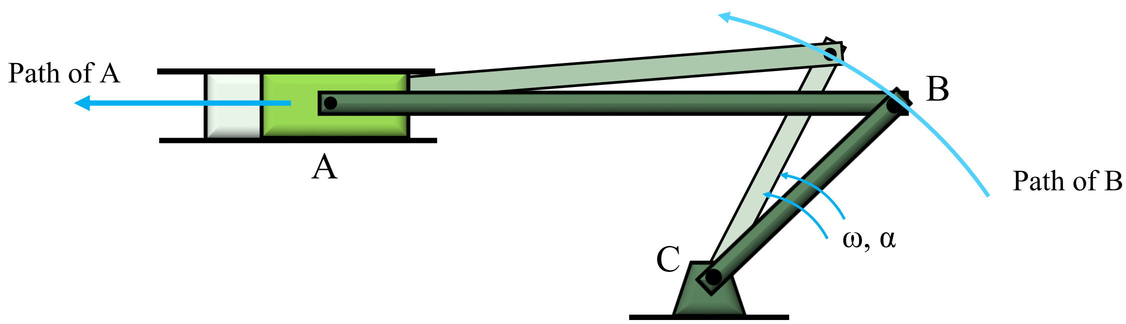


General Plane Motion Translation Rotation about point A

3. First we differentiate the relative velocity equation to obtain $\boldsymbol{a}_B = \boldsymbol{a}_A + \boldsymbol{a}_{B/A}$. Second, observe from the diagram above how $\boldsymbol{a}_{B/A}$ has both normal and tangential components. This allows us to write $\boldsymbol{a}_B = \boldsymbol{a}_A + (\boldsymbol{a}_{B/A})_\mathbf{t} + (\boldsymbol{a}_{B/A})_\mathbf{n}$. Substituting the known kinematic values for $(\boldsymbol{a}_{B/A})_t$ and $(\boldsymbol{a}_{B/A})_\mathbf{n}$ we obtain:

$$\boldsymbol{a}_B = \boldsymbol{a}_A + (\boldsymbol{\alpha} \times \boldsymbol{r}_{B/A}) - (\omega^2 \boldsymbol{r}_{B/A})$$

4. The relative acceleration equation is used to analyze the acceleration of various points on a body in a similar manner as with the relative velocity equation. We look at individual points on the body and their types of motion.



Application of relative acceleration equation to AB

Observe the mechanism above. Point B moves only in a circular path, meaning that we can calculate the acceleration at B as $\boldsymbol{a}_B = \alpha \boldsymbol{r}_{CB} + \omega^2 \boldsymbol{r}_{CB}$. Now look at point A, which can only move in the horizontal $\boldsymbol{i}$ direction, where the acceleration $\boldsymbol{a}_A$ is in the direction of motion. In general:

- For points that are constrained to move in a straight line, the acceleration must be directed along the path of motion
- For points tracing out a circular path, we know the tangential component of acceleration is $\alpha \boldsymbol{r}$, and the normal component of acceleration is $\omega^2 \boldsymbol{r}$

5. The procedure for analysis involves sketching the kinematic diagram for the general plane motion body and applying the relative acceleration equation to it. Since the equation requires ω, a relative velocity analysis may first be required.

Procedure for Analysis

i. Establish a fixed coordinate system

ii. Draw the kinematic diagram of the body.

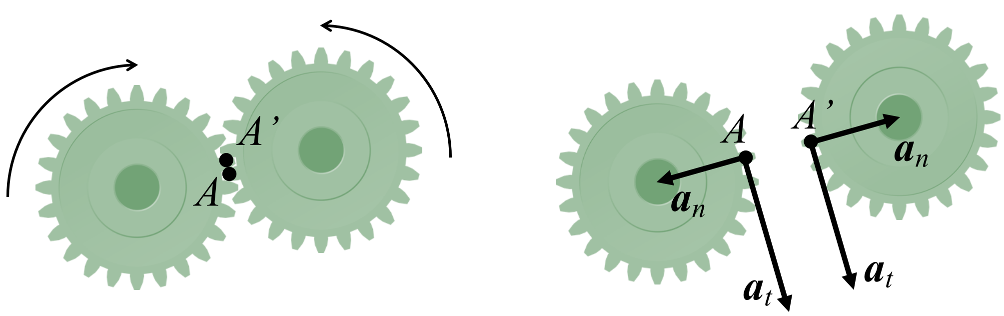
iii. On the diagram, indicate $\boldsymbol{a}_A$, $\boldsymbol{a}_B$, $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$, and $\boldsymbol{r}_{B/A}$. If the points A and B move along curved paths, then their accelerations should be shown in terms of their tangential and normal components.

iv. Apply the relative acceleration equation:

$$\boldsymbol{a}_{\boldsymbol{B}} = \boldsymbol{a}_{\boldsymbol{A}} + (\boldsymbol{\alpha} \text{ x } \boldsymbol{r}_{B/A}) - (\omega^2 \boldsymbol{r}_{B/A})$$

v. If the solution yields a negative answer for an unknown magnitude, then the actual direction of the vector is opposite of what was indicated on the kinematic diagram.

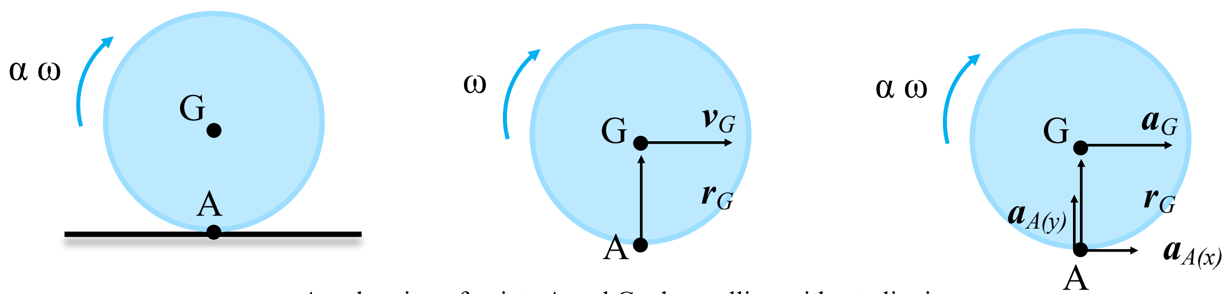
6. Certain conditions allows us to analyze bodies in contact, like the meshing of gears, as shown below.



Compatibility conditions of gears at mesh point

At the point of contact A (indicated as A' for the second gear), the tangential components of acceleration equate, meaning that $\alpha \boldsymbol{r}_{A'} = \alpha \boldsymbol{r}_A$. Note that the normal components do not necessarily equate.

7. Let us analyze rolling motion with respect to acceleration in cases where there is no slipping. In prior lessons, we learned that the point of contact with the ground is the instantaneous center of zero velocity. However, that point does not have a zero acceleration.



Acceleration of points A and G when rolling without slipping

8. Point G moves along a straight path to the right, so its acceleration is simply $\boldsymbol{a_G} = a_G \boldsymbol{i}$. At point A, consider the separate x and y components of its velocity just before and after touching the ground. The x component of velocity, and thus acceleration, continues in the same direction. The y component, however, switches from being oriented downward to oriented upward, meaning that the acceleration at A is in the upward direction, and $\boldsymbol{a_A} = a_A \boldsymbol{j}$. Using these two points, we can apply the relative acceleration equation:

$$\boldsymbol{a}_{\mathrm{G}} = \boldsymbol{a}_{\mathrm{A}} + \boldsymbol{\alpha} \times \boldsymbol{r}_{\mathrm{G/A}} - \omega^2 \boldsymbol{r}_{\mathrm{G/A}}$$

$$a_{\mathrm{G}} \boldsymbol{i} = a_{\mathrm{A}} \boldsymbol{j} + (\alpha\, \boldsymbol{k}) \times (r_{\mathrm{G/A}} \boldsymbol{j}) - \omega^2 r_{\mathrm{G/A}} \boldsymbol{j}$$

Evaluating and equating for $\boldsymbol{i}$ and $\boldsymbol{j}$ components:

$$\boldsymbol{a_G} = \alpha \boldsymbol{r} \text{ and } \boldsymbol{a_A} = \omega^2 \boldsymbol{r}$$

This will always be true for a wheel rolling without slipping.

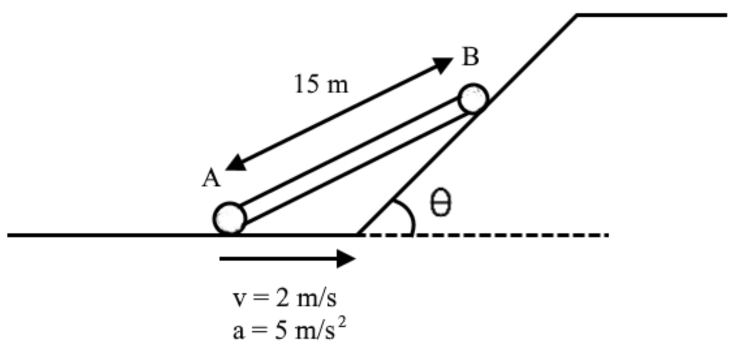
Example Problem

The rod AB moves along the plane as shown. Point A has an acceleration of a = 5 m/s^2 and a velocity of v = 2 m/s towards the right. Determine the angular acceleration of the rod.

Given: Dimensions of the rod. Acceleration and velocity of A.

Find: Angular acceleration of the rod.

Plan: Kinetic Diagram. Find angular velocity using center of zero velocity. Find angular acceleration using acceleration equations.

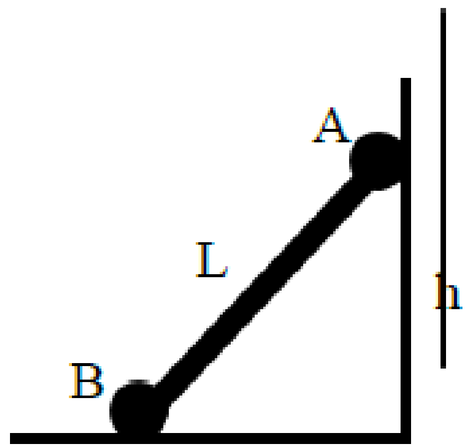


Lesson 22 Group Work

Given: At the instant shown, the rod has a height of h= 8m with the rod's length being L=10m. a_A = 3 m/s^2 and v_A= 7 m/s.

Find: The angular acceleration of the rod and the acceleration of B.

Plan:



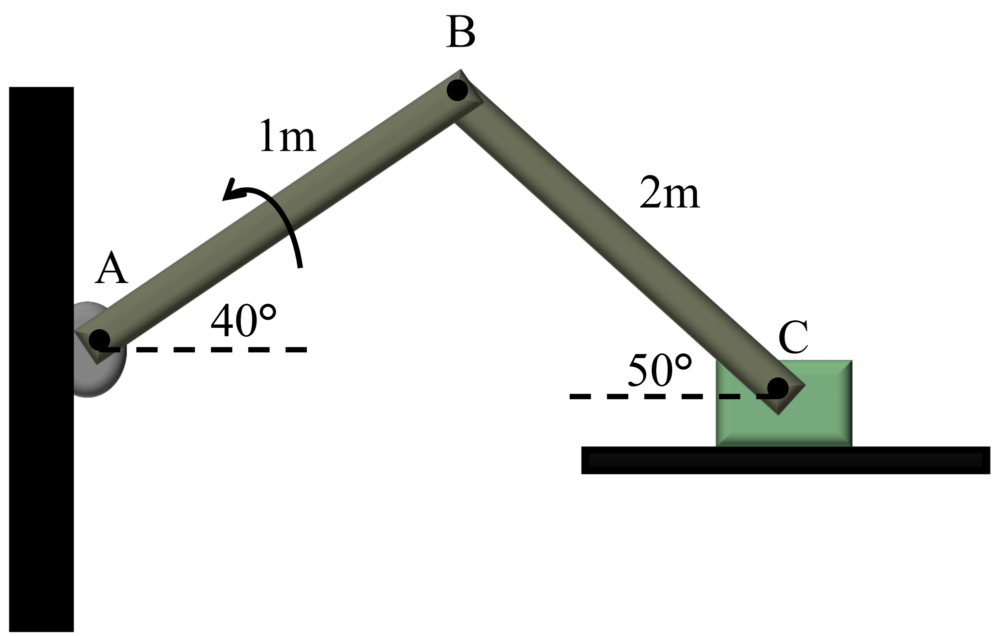
Angular Velocity:

Velocity of B:

Given: Calculate the angular acceleration of the plate in the position shown, where control link **AO** has a constant angular

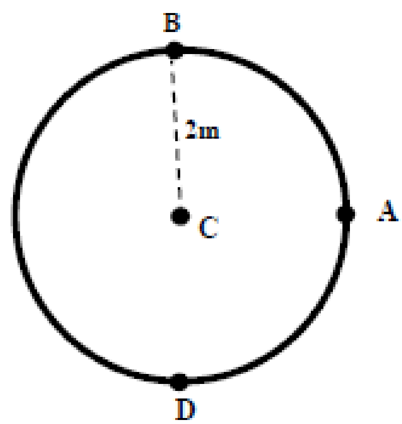
Homework Assignment # 22

Bar AB moves as shown with an angular velocity $\omega_{AB} = 5$ rad/s and angular acceleration $\alpha_{AB} = 7$ rad/s^2. Determine the velocity and acceleration of the sliding block C at this instant.



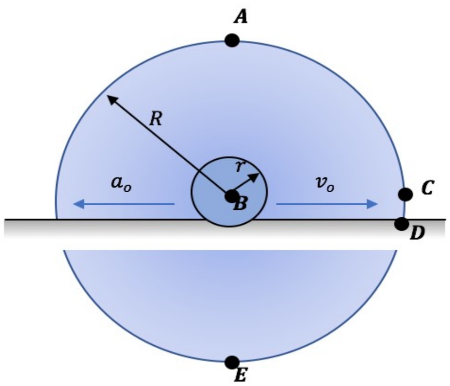
Lesson 22 Group Work [2]

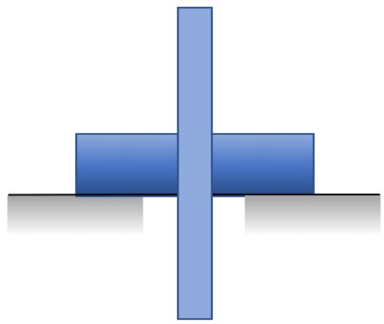
A history professor is running from an ancient stone cylinder which has an angular velocity ω=10 rad/s and angular acceleration α= 5 rad/s^2. Find the acceleration at points A and B.



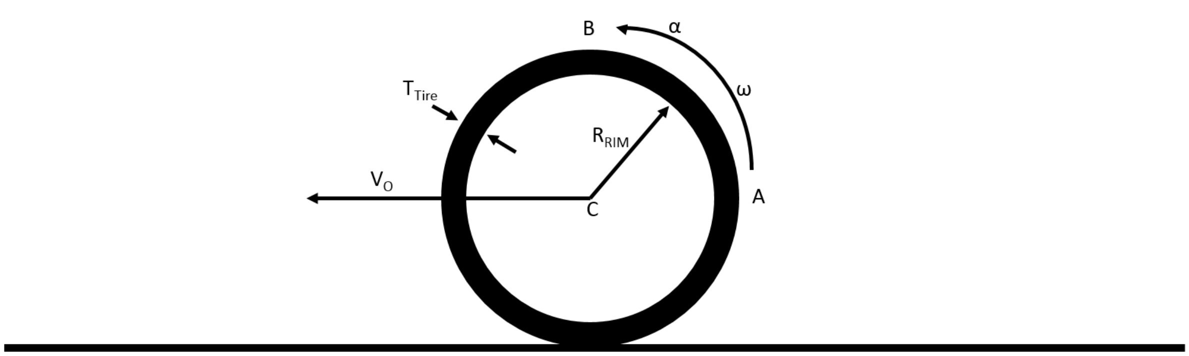
Homework Assignment # 22 [2]

1. The shaft of the wheel sits on a fixed horizontal surface. The wheel rolls without slipping and has an inner radius of 2 inches and an outer radius of 10 inches. The velocity of point B is 5 inches per second to the right and the acceleration of point B is 8 inches per second to the left. Determine the accelerations of both points A and D.





2. The Wheel of a Hennessey F5 has an angular acceleration of 5 radians per second per second when it is going 200 mph down the test track in West Texas. What is the Acceleration of points A and B if the Tire is 2 inches thick and the rim has a radius of 16 inches.

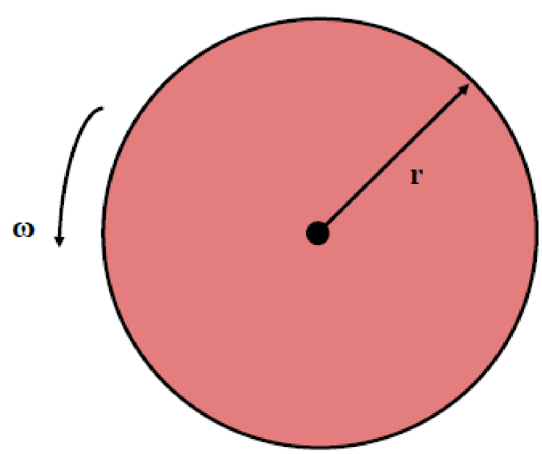


Mass Moment of Inertia

Lesson 23

1. A body's mass moment of inertia measures its resistance to rotational motion and is useful when analyzing the rotational motion of a rigid body. The mass moment of inertia shares similarities to the area moment of inertia; however, it assesses different properties and is almost exclusively used for rigid body dynamics problems.

2. Since the mass moment of inertia measures a body's resistance to rotational motion, it can be shown for a wheel, such as a flywheel in a motor, that the mass moment of inertia is greater for a wheel of larger radius. Once the large wheel is set in motion, it will be more difficult to stop its rotation, allowing the flywheel to maintain constant power and prevent stalling in an engine.

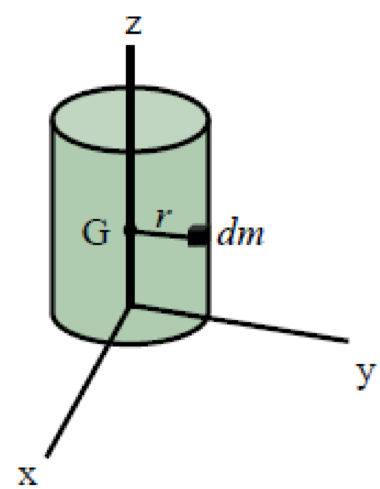


Rotating flywheel

3. A rigid body is free to rotate about an axis through its center of mass. Applying a moment or torque about this axis results in an angular acceleration of the body that is proportional to the applied torque. The constant of proportionality between torque and angular acceleration is the mass moment of inertia.

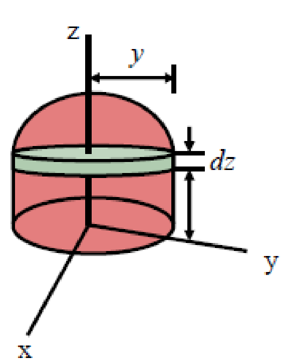
4. When finding the mass moment of inertia, we can identify an infinitesimal mass dm that is rotated using a moment arm of r, the perpendicular distance to the axis of interest. This axis generally passes through G, the center of gravity. Therefore, the mass moment of inertia of this cylinder can be computed along the z-axis as

$$\mathbf{I = \int r^2 dm}$$



Rotating body about z-axis

5. When considering an axisymmetric, three-dimensional rigid body, the mass moment of inertia of a simple geometric shape may be calculated as $\int r^2 dm$.The simple geometric shape may then be used as a differential element in integration to obtain the mass moment of inertia of the whole rigid body. For the shape shown, a thin disk is used in computing the mass moment of inertia along the z-axis through the center of gravity.



Axi-symmetric 3D solid

Parallel-Axis Theorem

The parallel axis theorem allows for the calculation of the moment of inertia about any axis parallel to a known axis. For example, if the mass moment of inertia is known about an axis passing through the center of gravity, then the mass moment of inertia about any axis parallel can be found using the following equation:

$$\mathbf{I = I_G + md^2}$$

By finding the mass moment of inertia of several simple shapes about their own center of gravity, the parallel axis theorem can be applied to find the total mass moment of inertia for a composite rigid body about its center of gravity.

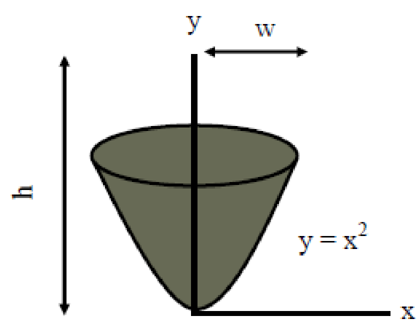
Radius of Gyration

The radius of gyration (k) is another way to define the mass moment of inertia of a body about a specific axis. The radius of gyration has units of length and measures of the distribution of mass about the axis in which the moment of inertia is defined. The two quantities are related by the following equations:

$$\mathbf{I = mk^2} \text{ or } \mathbf{k = (I/m)^{1/2}}$$

Example 1

A rigid body is created by revolving the curve $y = x^2$ about the y-axis. If the density of the material comprising the given rigid body is 2000 kg/m^3, calculate the mass moment of inertia about the y-axis. The height and width of the body formed are h = 1m and w = 1 m respectively.



Group Problem # 23

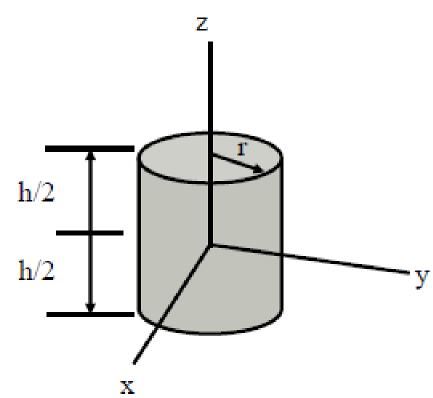
Determine the mass moment of inertia of the 2000 kg cylinder with the given dimensions. Assume the cylinder is a rigid body with constant density.

Given: $r = 2\text{m}$, $h = 5\text{m}$, $m = 2000$ kg

Find: Mass Moment of Inertia about the z-axis

Plan: Establish a circular disk element to use in integration

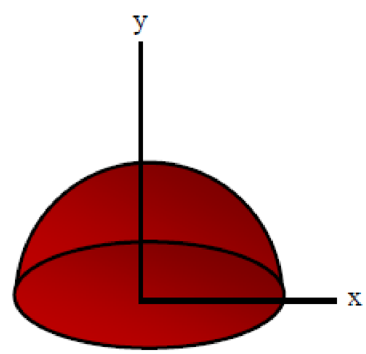
Set up an integral with the chosen differential element



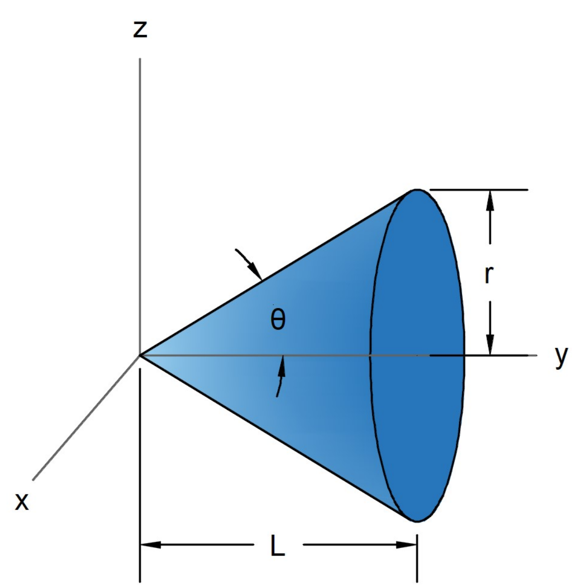
Integrate over the region of the cylinder to find the mass moment of inertia

Homework Assignment # 23

1. The top half of a sphere is created by rotating the circle $x^2 + y^2 = r^2$ around the y-axis. If the rigid body has mass m and a constant density ρ, find the mass moment of inertia about the y-axis in terms of total mass and density.



2. Calculate the mass moment of inertia about the y-axis of the cone shown below. The cone is made of a uniform material of density 2.5 kg/m. The total length of the cone is of 5 m. Also, note that $\theta = 35^{o}$.



Group Problem # 23 [2]

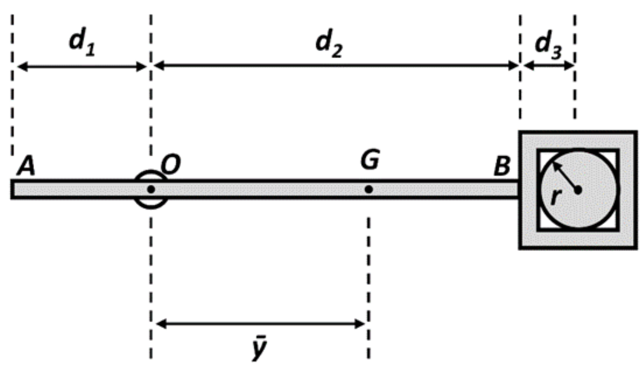
Given: An assembly of rod AB, with a mass of 2 kg/m, and the thin plates attached at B, with a mass of 8 kg/m^2. The distances $\boldsymbol{d_1}$, $\boldsymbol{d_2}$, and $\boldsymbol{d_3}$ are 0.5 m, 2.0 m, and 0.3 m, respectively. The radius of the circular plate inside of the square frame, $\boldsymbol{r}$, is 0.2 m.

Find:

a) The location of the center of mass G of the assembly

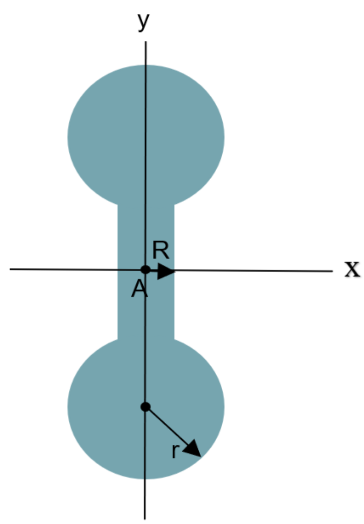
b) The moment of inertia about an axis through O, perpendicular to the page.

c) The moment of inertia about an axis through G, perpendicular to the page.

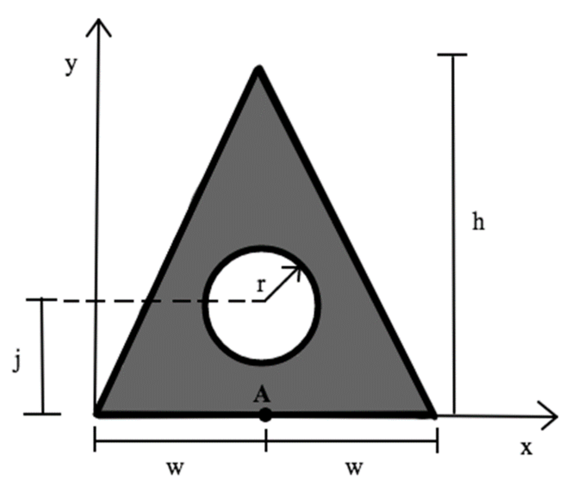


Homework Assignment # 23 [2]

1. The dumbbell at the right can be broken down into two spheres of the same size with a mass of 4 kg each and a radius r of 5 cm. The bar in between is a rod with radius R of 2.5 cm and a mass of 2 kg. a) Find the mass moment of inertia through the center of mass at A. b) Find the mass moment of inertia through the axis B.



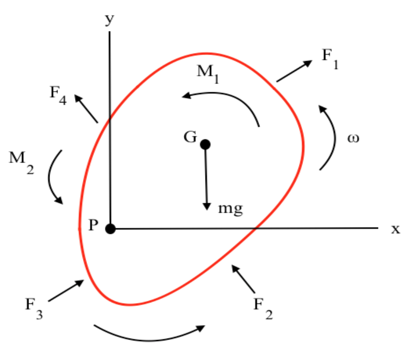
2. Determine the mass moment of inertia of the thin plate shown below about an axis perpendicular to the page and passing through point A. The material has a mass per unit area of 35 kg/m^2. The dimensions are as follows: r = 0.5m, h = 6m, w = 2m, j = 2m.



Planar Equations of Motion-Translation

Lesson 24

1. Rigid bodies have the ability to rotate as well as translate. We will apply the equations of motion to determine how both forces and moments cause bodies to accelerate. We will look first at the case of translation, then pure rotation, and finally superimpose those two results to analyze the general plane motion kinetics of a rigid body.



Rigid body

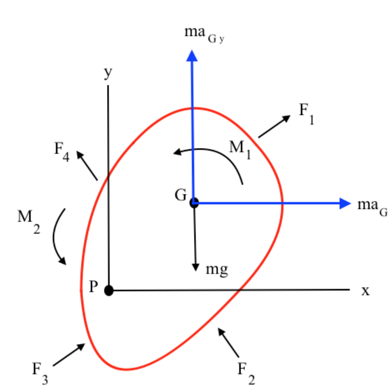
2. A rigid body is shown subjected to a series of externally applied forces and couple moments. A free body diagram is necessary in order to determine all the externally applied forces present on the body.

3. We may extend Newton's second law to describe the translational motion of a rigid body. For a particle, we showed the governing equation to be $\sum \mathbf{F} = m\mathbf{a}$. For a rigid body, we apply Newton's second law to the system of particles comprising the rigid body. Since all internal forces cancel, the alternative version of Newton's second law for rigid bodies is:

$$\sum \mathbf{F} = m\mathbf{a_G}$$

where $\mathbf{a_G}$ is the acceleration of the center of mass of the body.

We may extract two scalar equations from this vector equation, representing the two coordinate directions.



FBD and Kinetic Diagram

4. We need to determine the effects caused by the moments of an external force system. We may select any convenient point P on both the free body and kinetic diagrams. The rotational equation of motion in its most general form is:

$$\sum \mathbf{M_P} = \sum(\mathbf{M_K})_\mathbf{P}$$

The sum of the moments about point P of all the externally applied forces and couple moments shown on the free body diagram is equal to the sum of the moments about point P of the inertial forces shown on the kinetic diagram.

5. If we choose point P to be the center of mass G, then the rotational equation of motion reduces to

$$\sum \mathbf{M_G} = I_G \boldsymbol{\alpha}$$

Thus, three independent scalar equations of motion may be used to describe the general planar motion of a rigid body. These equations are:

$$\sum \mathbf{F_x} = m\mathbf{a_{Gx}}$$

$$\sum \mathbf{F_y} = m\mathbf{a_{Gy}}$$

$$\sum \mathbf{M_P} = \sum(\mathbf{M_K})_\mathbf{P}$$

6. For the case of rigid body translation only, there is no rotation and $\boldsymbol{\alpha} = 0$. The kinetic diagram will not contain an $I_G\boldsymbol{\alpha}$ term, just the translational inertia forces. The same three equations of motion may be used, and if the center of mass is chosen as point P, the third equation becomes $\sum \mathbf{M_G} = 0$.

7. When a rigid body is subjected to curvilinear translation, is is best to use an n-t coordinate system. Then apply the equations of motion, as written below, for the n-t coordinates:

$$\sum \mathbf{F_n} = m\mathbf{a_{Gn}}$$

$$\sum \mathbf{F_t} = m\mathbf{a_{Gt}}$$

$$\sum \mathbf{M_G} = \mathbf{0} \text{ or } \sum \mathbf{M_P} = \sum (\mathbf{M_K})_\mathbf{P}$$

8. The procedure for solving rigid body translation problems is as follows:

i. Establish an (x-y) or (n-t) inertial coordinate system and specify the sense and direction of acceleration of the mass center $\mathbf{a_G}$.

ii. Draw a FBD and kinetic diagram showing all external forces, couples and the inertial forces and couples.

iii. Identify the unknowns.

iv. Apply the three equations of motion

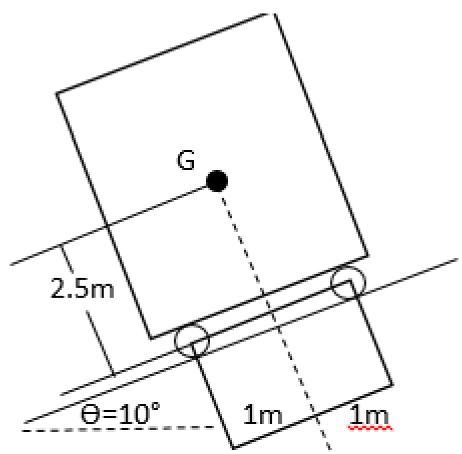
v. Remember, friction forces always act on the body opposing the motion of the body.

Example 1

Given: Determine the acceleration of the 95-kg cabinet as it rolls down the inclined plane and the normal reactions on the pair of rollers at A and B, that have negligible mass.

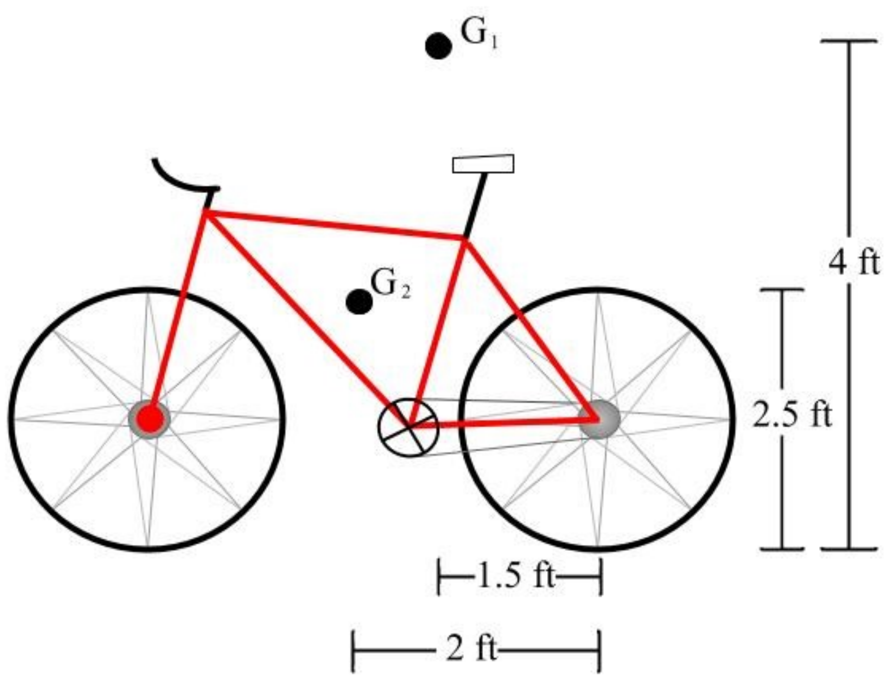
Find: $a = ?$ $N_A = ?$ $N_B = ?$

Plan: Draw FBD and Kinetic Diagram, apply equation of motion in x-direction to get a, apply equation of motion in y-direction to get Eq (1), take the moment about G to get Eq (2), solve equation (1) and (2) simultaneously to get N_A and N_B.



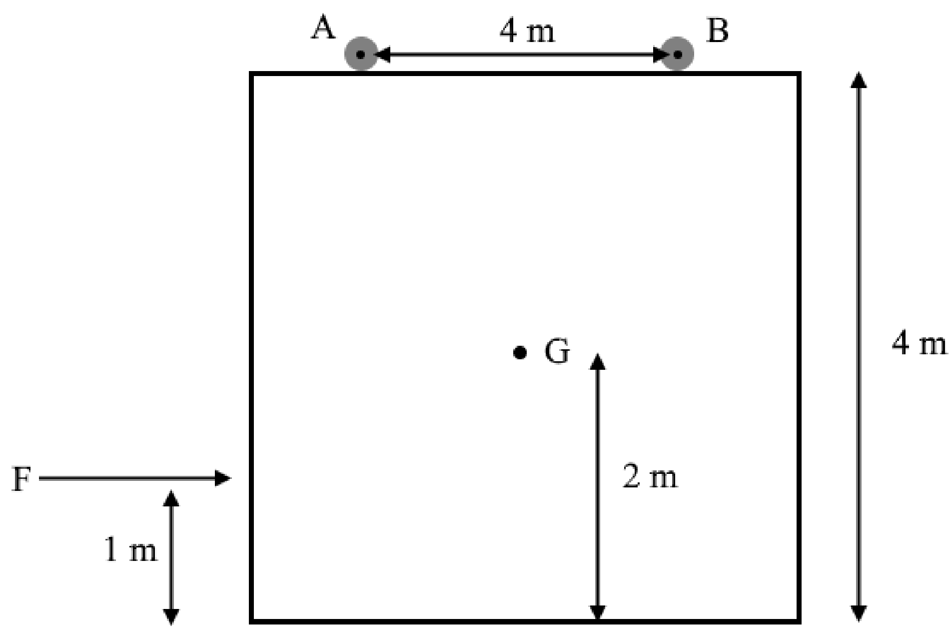
Group Problem # 24

A 180 lb rider, centered at G1, is riding a 35 lb bicycle, centered at G2. Given the distances shown in the diagram, find the acceleration required to bring the front wheel off the ground. Also, find the traction force and normal reaction under the rear wheel.



Homework Assignment # 24

1.The door has a mass of 82 kg and a center of gravity at G. Determine the constant force F that must be applied to push the door open 4 m to the right in 6 seconds, starting from rest. In addition, find the vertical reaction of rollers A and B.



Homework Assignment # 24

2. The Hennessey Venom F5 has a weight of 2950 lb. The car starts from rest spinning its wheels (slipping) down the test track in west Texas. The Michelin tires have a coefficient of static friction of μ_s = .5 and a coefficient of kinetic friction of μ_k = 0.3 with the test track. How long will it take to reach its top speed of 300 mph and what will the normal forces be acting up from the road on each tire.

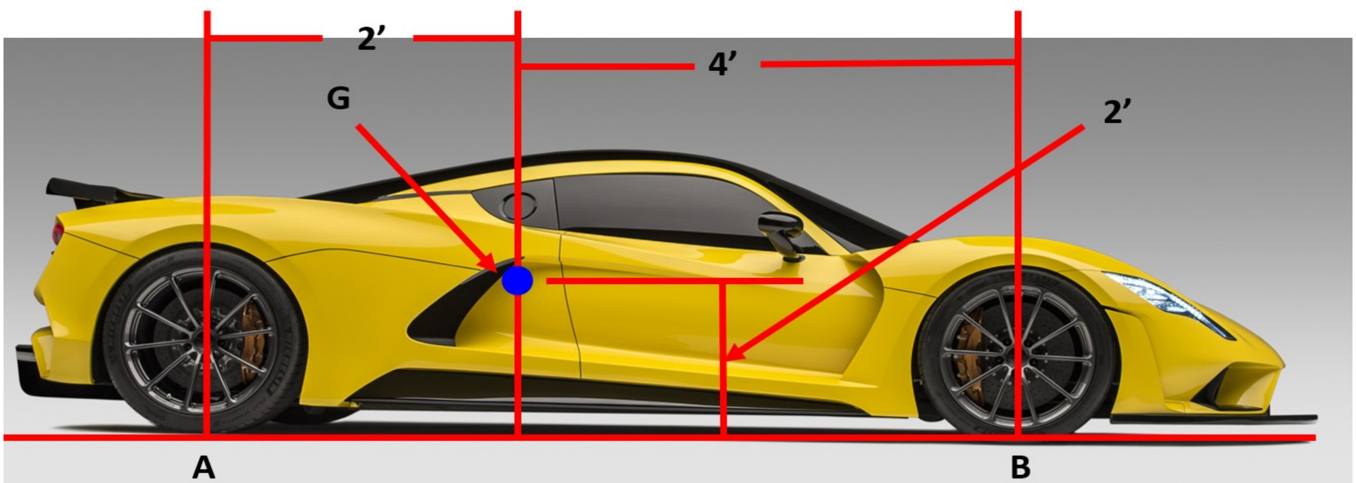


Figure 1: In the image above, the new Hennessey Venom F5 is pictured capable of reaching 300+ mph. It is powered by a 7.4 L twin turbocharged V8 producing 1600 BHP. Image courtesy of http://www.hennesseyspecialvehicles.com/.

Example 2

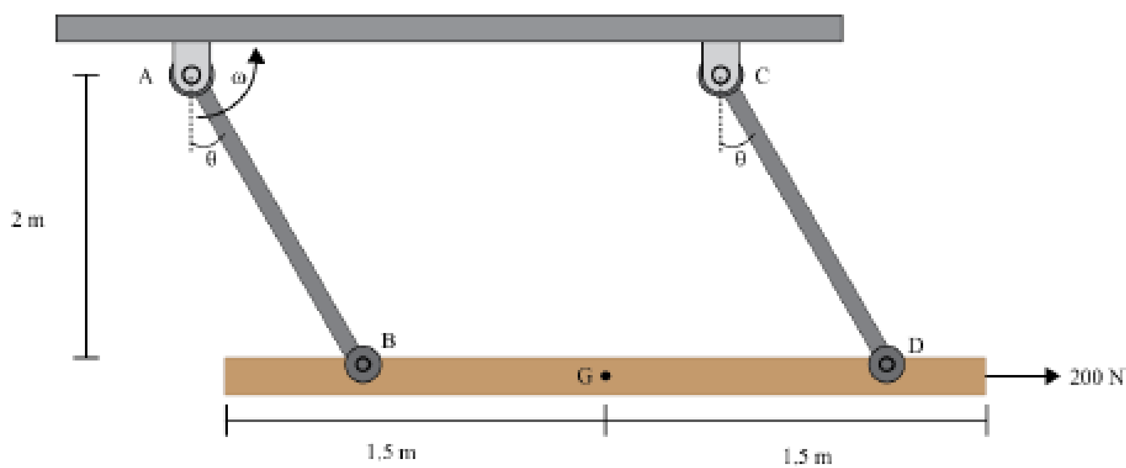
Given: A 5-kg bar shown in the figure is swinging with an angular velocity of 3 rad/sec counterclockwise. The 3-m bar is pushed by a 200 N horizontal force.

Find: The tension force in the rods and the angular acceleration of the system at the instant $\theta = 30^0$.

Plan: Draw a FBD and kinetic diagram

Apply the equations of motion

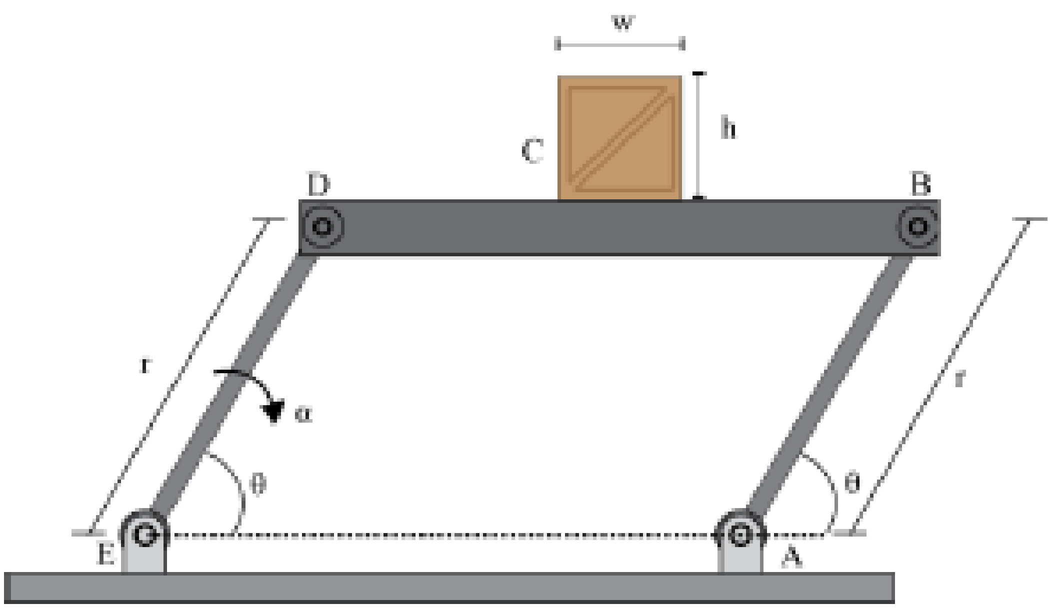
Solve for T and α



Group Problem # 24 [2]

1. **Given:** Box C, of mass m rests on the horizontal bar with a coefficient of static friction of $\mu_s = 0.3$. Assume no tipping occurs.

Find: The largest angular acceleration from rest at $\theta = 45^0$ without the box slipping.



Plan: Draw a FBD and kinetic diagram of the box

Group Problem # 24 [2]

Calculate the normal and tangential acceleration of the box at $\theta = 45^0$.

Write the normal equation of motion to calculate the normal reaction.

Write a moment equation to find the distance from the center of gravity that the normal force acts on the box.

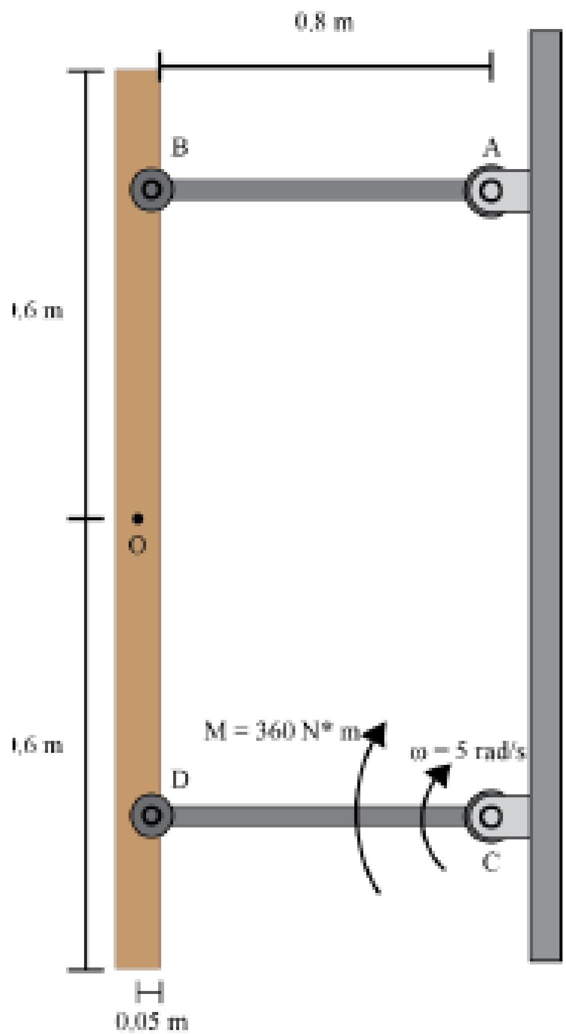
Write the tangential equation of motion to solve for α.

Substitute these values:

$\omega = 5$ rad/sec, $r = 2$m

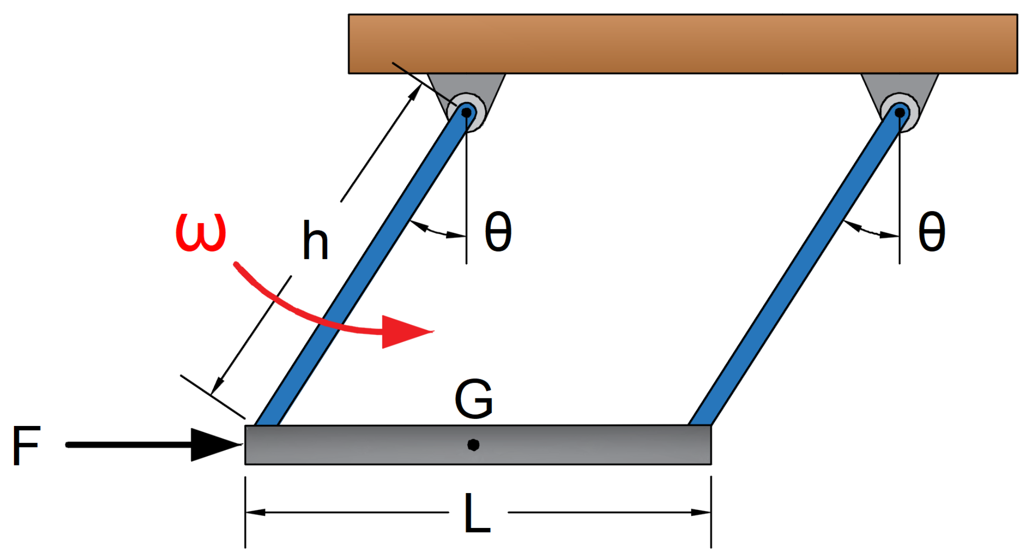
Homework Assignment # 24 [2]

1. A 1.2m vertical bar has a mass of 40 kg and a center of mass at O. It is connected to a wall by links AB and CD, each of which has a length of 0.8m. At the instant shown, link CD rotates with a positive angular velocity of ω = 5 rad/sec. Link CD is also subjected to a clockwise 360 N-m couple moment. Determine the tensile force in link AB, the angular acceleration of link CD, and the horizontal and vertical reaction components at pin D. Neglect the mass of links AB and CD.



Homework Assignment # 24 [2]

2. At the moment shown, both of the rods of negligible mass move with an angular velocity $\omega = 2$ rad/s and h=4m. At the same time, the 20 kg bar with a length of 3m experiences a horizontal force of 50 N. Determine the angular acceleration and the tension in the two rods at this instant if $\theta = 30°$.

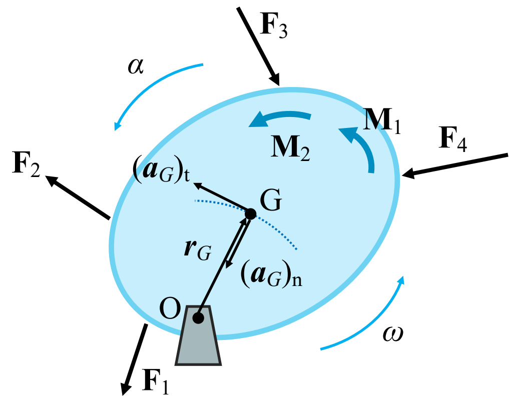


Rotation about a Fixed Axis

Lesson 25

1. Having dealt with the translation of rigid bodies, let's move on to look at the kinetics of rigid body **rotation**.

2. Observe the diagram at right. Recall how for any rigid body rotating about a point O, the center of mass G follows a circular path (as does any point on the body). Therefore, G has acceleration components of αr in the tangential direction, and $\omega^2 r$ in the normal direction. From the acceleration components, we can write three equations of motion:



Rigid body rotating about O

$$\sum \mathbf{F_n} = \text{m}(\boldsymbol{a}_{\boldsymbol{G}})_\mathbf{n} = \text{m}\boldsymbol{r}_{\boldsymbol{G}}\boldsymbol{\omega}^2 \quad \sum \mathbf{F_t} = \text{m}(\boldsymbol{a}_{\boldsymbol{G}})_\mathbf{t} = \text{m}\boldsymbol{r}_{\boldsymbol{G}}\boldsymbol{\alpha}$$

$$\sum \mathbf{M}_{\boldsymbol{G}} = \text{I}_G\boldsymbol{\alpha}$$

3. We don't have to chose G as the point about which to sum moments. We can chose any alternative point, and for fixed axis rotation, it can be convenient to sum moments about the point of rotation O. This yields $\sum M_O = I_G\,\alpha + r_G\,m(a_G)_t = [I_G + m(r_G)^2]\alpha$. Observe how the latter part of the equation is the parallel axis theorem for point O. We can now write three alternative equations of motion about point O:

$$\sum \mathbf{F_n} = \text{m}(\boldsymbol{a}_\mathbf{G})_\mathbf{n} = \text{m}\boldsymbol{r}_\mathbf{G}\boldsymbol{\omega}^2 \quad \sum \mathbf{F_t} = \text{m}(\boldsymbol{a}_\mathbf{G})_\mathbf{t} = \text{m}\boldsymbol{r}_\mathbf{G}\boldsymbol{\alpha}$$

$$\sum \mathbf{M_O} = \text{I}_\text{O}\boldsymbol{\alpha}$$

4. After determining these kinematic relationships, the procedure for solving problems with rigid body rotation is the same as solving problems of translation:

i. Establish an inertial coordinate system and specify the sense and direction of acceleration of $(\mathbf{a}_G)_n$ and $(\mathbf{a}_G)_t$.

ii. Draw a Free Body Diagram with all external forces and couple moments. Then draw the Kinetic Diagram showing the inertial forces and couple moment.

iii. Compute the mass moment of inertia I_G or I_O.

iv. Write the three equations of motion and identify the unknowns. Solve for the unknowns.

v. Use kinematics if there are more than three unknowns, since the equations of motion allow for only three unknowns. This will often arise in problems involving friction.

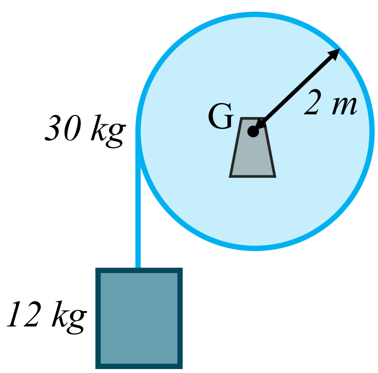
Example 1

A solid wheel with uniform mass of 30kg has a pin connection in the middle of the wheel. The radius is 2m and its radius of gyration is 1m. A rope is wrapped around the wheel with a block of 12kg attached to the end. When the block is released from rest, find the angular acceleration of the wheel.

Given: Mass of wheel 30kg, rope wrap around wheel with block hanging of 12kg.

Find: The angular acceleration of the wheel.

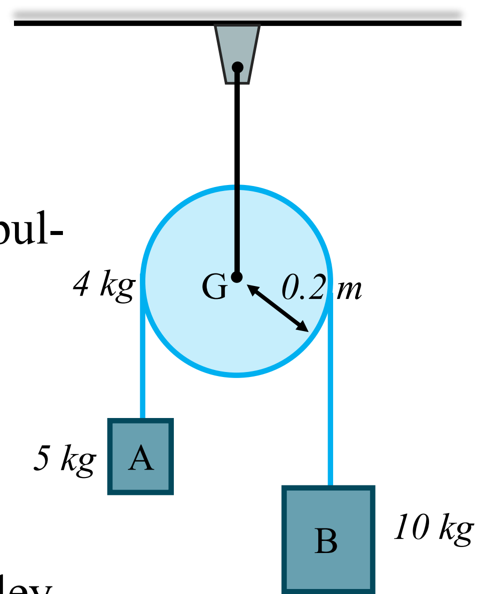
Plan: Draw FDB of the wheel and the block. Find I_G by using $I_G=m(k_G)^2$. Find tension T by using summation of moment M_G. Substitute in T for summation of F_y. Substitute in a for $a = \alpha r$, then solve for α.



Group Work # 25

The two blocks A and B have a mass of 5 kg and 10 kg, respectively. If the pulley can be treated as a disk of mass 4 kg and radius 0.2 m, determine the acceleration of block A. Neglect the mass of the cord and any slipping on the pulley.

Plan:



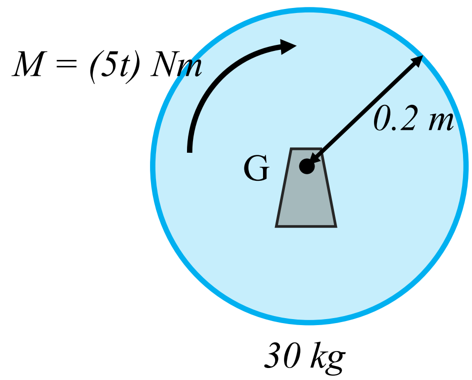
Relate the angular acceleration of the pulley to the acceleration of block A:

The mass moment of inertia of the pulley about point G:

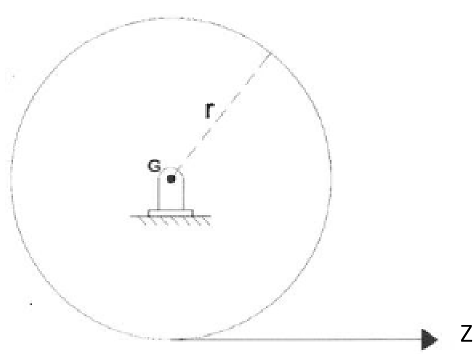
Use moment equation of motion to find acceleration of block **A**:

Homework Assignment # 25

1. The 30 kg disk experiences a couple moment of M=(5t) Nm, where t is in seconds. Find the angular velocity ω of the disk when t=3s, beginning from rest.

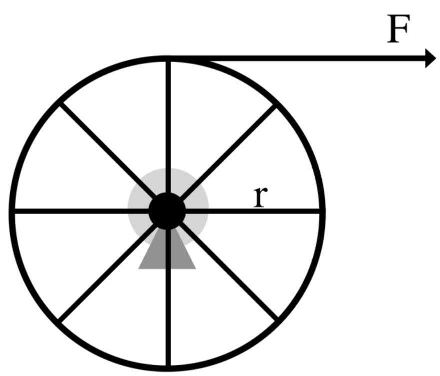


2. The wheel in the diagram has a radius r = 0.2m and rotates about its center of mass at G. The wheel's radius of gyration is k_G = 0.8m and its mass is 200 kg. The force acting at the bottom of the wheel is Z = ($16t^3$ + 40) N, where t is in seconds. Find the angular velocity of the wheel at t = 2 seconds, if the wheel starts from rest.



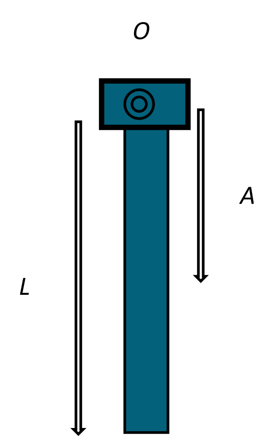
Example 2

1. A force of F=7lb is applied to the cord attached to the pulley. The pulley has a weight of 15lb, and a radius of r = 0.4ft. The radius of gyration for the pulley is k_G=0.3ft. Determine the acceleration of the pulley, and the tangential acceleration of the rope. You can assume friction in the pin is negligible.



Group Work # 25 [2]

The most popular game at a local fair comprises of a vertical rod target hanging from a singular pin mounted to the roof of the stall. The rod is 1.25 meters long, and has a mass of 2kg. The player is given an air rifle, for a low price $2.00 that shoots a pellet with a force of the players choosing to start at 10N with each additional 10 N force costing an additional 25 cents up till 100N. To win the game the target must be displaced from its pinned location, knowing that the pin exerts a horizontal force of 14.8 N, determine the force applied by the pellet angular acceleration of the rod about the pin if the force applied by the pellet was 25 N at point A, one meter from the pin. If the required angular acceleration to win is 18 rad/s^2, did this pellet result in a win; if not calculate the minimum force required for a win and the cost to win.



Group Work # 25 [2]

Draw the FBD and KD for the rod.

Equation of Motion about Point O.

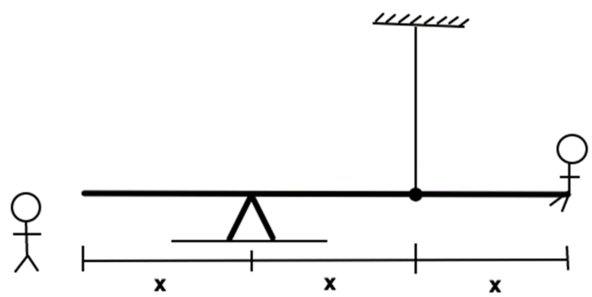
Using Equations of Motion solve for α.

Check the win requirement.

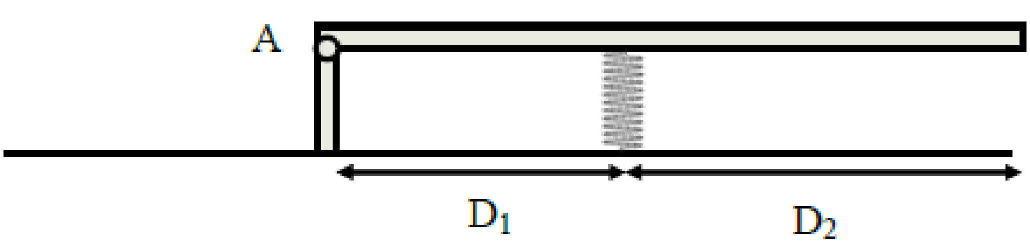
Calculate new force and cost for a win.

Homework Assignment # 25 [2]

1. Two people, both with masses of 150 kg, are on an uneven seesaw at the instant that it is parallel to the ground. The seesaw has an elastic cord with spring constant $k = 200$ N/m tied in the center of the longer side. The cord is stretched 0.3 m. The person opposite the cord gets off making the other person (still on their seat) fall to the ground. Find the reaction forces at the pin and the angular acceleration of the seesaw at the instant the person gets off. $\omega = 0$, $m = 50$ kg, $x = 1$ m.



2. A woman jumps off of a 30-kg diving board. Assuming that the board is uniform, find the angular acceleration of the board and the horizontal and vertical components of reaction at pin A, at the instant she jumps off. The spring is compressed 0.25m and the board has an initial angular velocity of 0. the spring constant is $k = 6.5$ kN/m. $D_1 = 1$m and $D_2 = 3$m.



General Plane Motion

Lesson 26

1. Let's complete our study of rigid body kinetics by the direct application of the equations of motion by looking at general plane motion of rigid bodies.

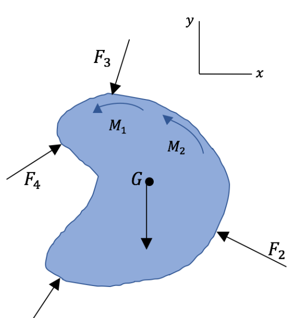
Lawn roller—general plane motion

2. This lawn roller is an example of a rigid body undergoing both translation and rotation. We would like to be able to determine the translation and angular acceleration of the roller as a result of being subjected to external forces, including the effect of friction.

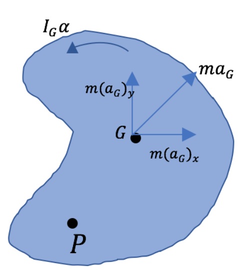
3. Unbalanced forces and moments induce rigid bodies to both translate and rotate. This combination is called general plane motion. Using an x-y inertial coordinate system, the equation for motions about the center of mass, G, may be written as:

$$\sum F_x = m(a_G)_x \quad \sum F_y = m(a_G)_y \quad \sum M_G = I_G\alpha$$

The free body diagram is drawn showing all externally applied forces and couple moments. The kinetic diagram is drawn showing the two components of translational inertial forces, and the rotational inertial moment $I_G\alpha$.



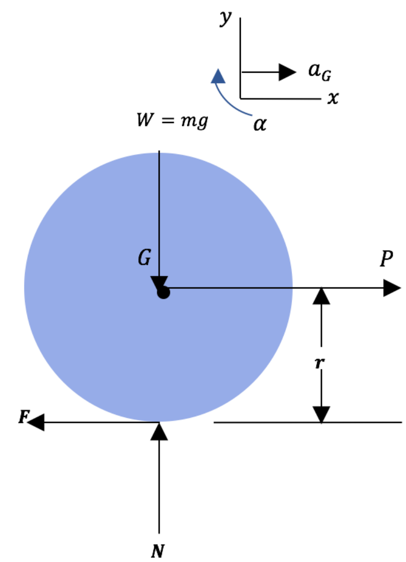
FBD



Kinetic Diagram

4. Once again, it may be more convenient to express the moment equation about some point other than the center of mass. In which case the moment equation becomes $\Sigma M_P = \Sigma\,(M_k)_P$, where the right hand side of the equation represents the moments of $I_G\alpha$ and ma_G about point P.

5. A common type of rigid body general plane motion is rolling. If rolling occurs without slip between the disk and the ground, we model the friction as a static friction force. If slipping occurs, we model the friction as a kinetic friction force. In some cases we do not know if the disk will slip, and we are left with more unknowns than we have equations, and we must make assumptions to solve the problem. After solving the problem, we must then verify that our assumption was correct.



FBD of rolling disk

Two Assumptions

<u>No slip</u>: the contact point becomes a point of instantaneous zero velocity, and we remember from kinematics that $\mathbf{a_G = \alpha r}$. To verify, $F_f \leq \mu_s N$.

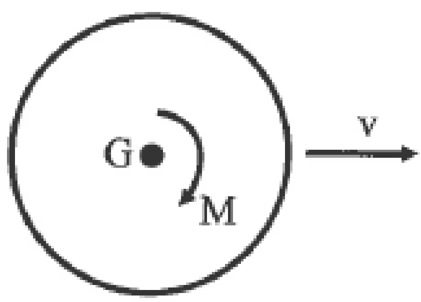
<u>Slipping</u>: we know that $\mathbf{F_f = \mu_k N}$. To verify, the direction of the force of friction must oppose the motion of the object.

6. With those unique aspects of general plane motion included, the procedure for solving kinetic problems follows the same steps. Draw a free body diagram, draw a kinetic diagram, and apply the three equations of motion.

7. Determine the number of unknowns. If necessary, make a friction assumption, solve the problem and check the validity of assumption. If sliding is assumed, the direction of the friction force must oppose the motion. If no-slip occurs, the direction of the friction force sometimes isn't as apparent. The friction force direction in rolling problems needs to be consistent with the assumed direction of α. In some cases, it is convenient to sum moments about the contact point, in which case the friction force does not enter into the equation.

Example 1

A wheel with mass of m = 50kg, radius r = 1m and radius of gyration of k_G = .35m rolls as shown. The coefficients of friction for this surface are $\mu_s = 0.4$ and $\mu_k = 0.2$. If the applied couple moment is M = 150 N-m, find the linear acceleration of the wheel.



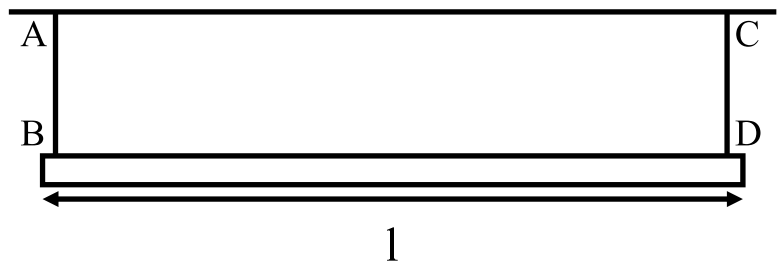
Plan: Draw a free body diagram and kinetic diagram for the wheel. Apply the equations of motion and frictional assumption to solve for unknowns

Group Problem # 26

A bar with mass m = 10kg and length l = .5m is held by two vertical cables. If cable CD breaks, releasing the bar, determine the tension in cable AB at that moment.

Given: m = 10kg, l= .5m

Find: T_{AB}



Plan: Draw Free Body and Kinetic Diagrams:

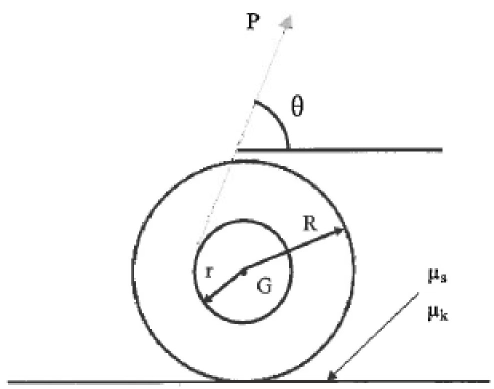
Write equations of motion:

Find acceleration components using relative acceleration:

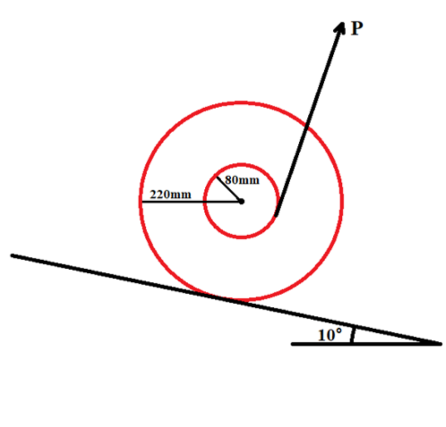
Solve for α, a_G, and T_{AB}

Homework Assignment # 26

1. A rope runs around a circular passage of radius r = 250 mm in a spool of radius R = 1000 mm. The mass of the spool is m = 500 kg and its centroidal radius of gyration is 200 mm. A constant tensile force of P = 400 N is applied on the rope at an angle $\theta = 30^0$. Calculate the angular acceleration of the spool, acceleration of its mass center G, and the frictional force between the floor and the spool. Assume $\mu_s = 0.4$ and $\mu_k = 0.2$.

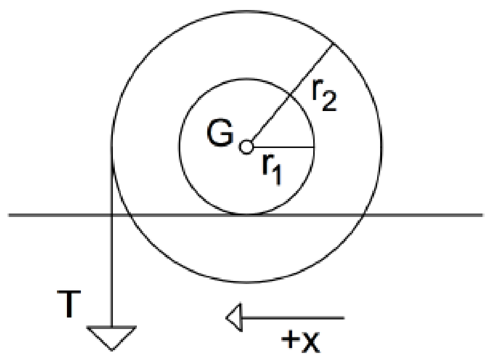


2. A 10 kg wheel is pulled up a small incline by a rope of force P = 20 N perpendicular to the motion. The wheel has a radius of gyration of k = 160mm. Find the angular acceleration of the wheel, the acceleration of its center of mass, and the force of friction. Assume $\mu_s = 0.2$ and $\mu_k = 0.15$



Example 2

A spool of mass 70 kg has a radius of gyration k = 300 mm about its center of mass. The inner radius of the spool is 400 mm and the outer radius is 800 mm. A vertical force T = 250 N is applied to the spool by a cable. Find the angular acceleration of the spool and the linear acceleration of its center of mass given that the spool slips and $\mu_k = 0.25$.

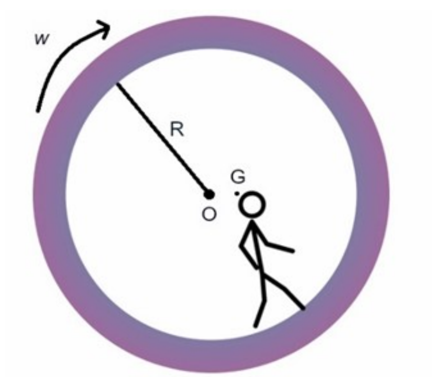


Group Problem # 26 [2]

Given: The hollow tube rolls with an angular velocity of $\omega = 0.75$ rad/s when the child is at the position shown. At this instant, the center of gravity of the tube and child is located at point G, and the radius of gyration about G is $k_G = 2.2$ ft. The combined weight of the tube and the child is 110 lb. Assume that the tube rolls without slipping, and the child does not move within the tube. R= 3ft, and the distance between O and G is 0.5 ft

Find: The acceleration of *G* and the angular acceleration of the tube.

Plan: Equations of motion: General Plane Motion.



Group Problem # 26 [2]

Plan: Equations of motion: General Plane Motion.

Determine I_G of the tube:

Draw FBD and Kinetic Diagram

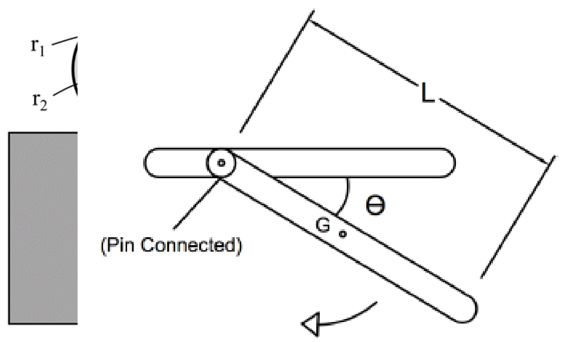
Use kinematic equations to solve for a_G

Find moment of inertia about G

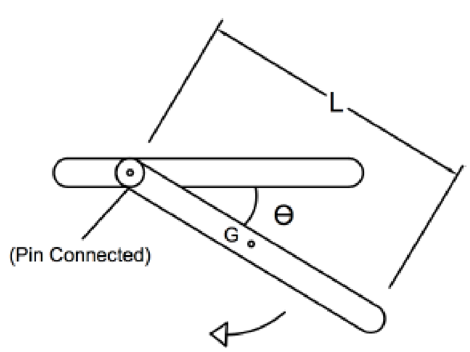
Use $a_{ICx} = 0$ to solve for a_O, then equate moments using $\Sigma M_{IC} = \Sigma(M_k)_{IC}$

Homework Assignment # 26 [2]

1. A spool or wheel of mass m_1 = 175 kg with a radius of gyration k_G = 0.2 m is attached by a cable to a frictionless pulley. The radii of the spool are r_1= 0.2m and r_2 = 0.5m. The coefficients of kinetic and static friction at B are μ_k = 0.2 and μ_s = 0.3, respectively. Determine the angular acceleration of the spool if a block of mass m_2 = 75 kg is attached to the end of the cable.



2. A slender rod hangs with one end attached to a roller in a horizontal slot. The mass of the rod is 15 kg, and the rod is 0.75 m long. If the rod is originally held at a horizontal and dropped, what are the x and y components of the pin reaction and the angular acceleration of the rod?



Work Energy for Rigid Bodies

Lesson 27

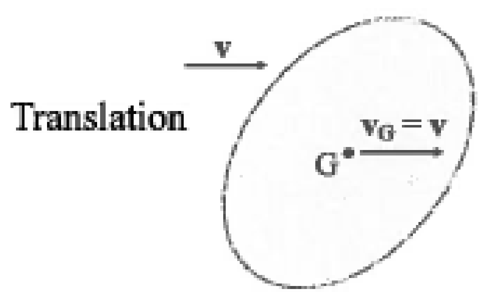
1. Let us re-visit the principle of work and energy, for the purposes of applying it to rigid body kinetic problems. Once again, the primary addition will be to accommodate the body's rotational motion, as well as its translation. This will affect the formulation of kinetic energy, and add a rotational work term.

2. First, we **expand our expression for kinetic energy to include rotational**, as well as translational kinetic energy. The general expression becomes

$$\mathbf{T = (1/2)mv_G^2 + (1/2)I_G\omega^2}$$

3. For the case of pure translation, the kinetic energy becomes a function of the velocity of the center of mass, and we obtain an expression identical to that of a particle,

$$T = (1/2)mv_G^2.$$

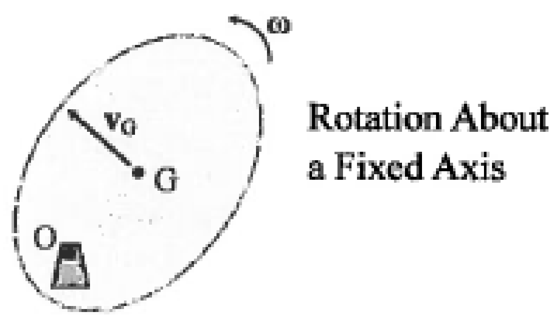


Kinetic energy of translating body

4. For the case of fixed-axis rotation about a point other than the center of mass, the center of mass has a velocity, and there is an angular velocity of the body, so both kinetic energy terms are necessary. Since $v_G = r_G\omega$, we can express the kinetic energy of the body as

$$T = (1/2)I_O\omega^2$$

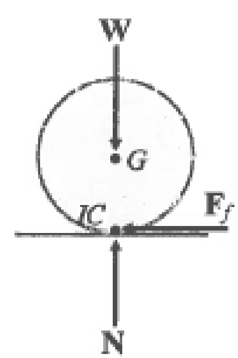
where O is the point of rotation.



Kinetic energy of rotating body

5. The general formulation of the expression for the work done by a force remains the same. This can be extended to indicate that work done by a weight and work done by spring forces are the same as the work done on a particle.

6. Recall that there are several forces that do no work because there must be a displacement in the direction of the force. In the diagram it can be seen that the weight, the normal force, and the friction force produce no work. Remember the friction force always acts at the contact point, which is not moving at that instant.



Forces doing no work

7. We do have to account for the fact that as the rigid body rotates, any externally applied couple moments produce work, if they act through the rotation. The work done by a moment is calculated by multiplying the moment by the angular displacement through which it acts.

$$U_M = \int M d\theta$$

Work is positive if M and θ are in the same direction.

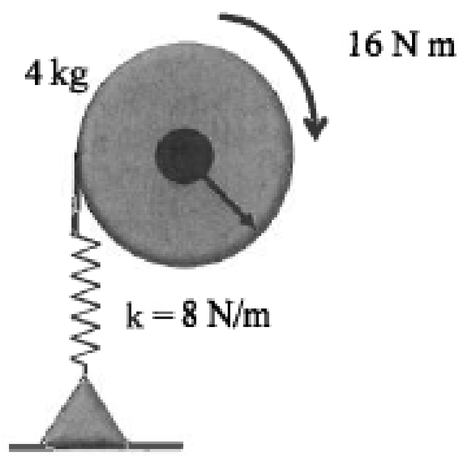
8. With the addition of rotational kinetic energy terms, and the work produced by couple moments, the principle of work and energy takes the identical form as was used for particles.

$$T_1 + \sum U_{1\text{-}2} = T_2$$

This is a helpful approach for solving kinetic problems that deal with forces, velocities and distances.

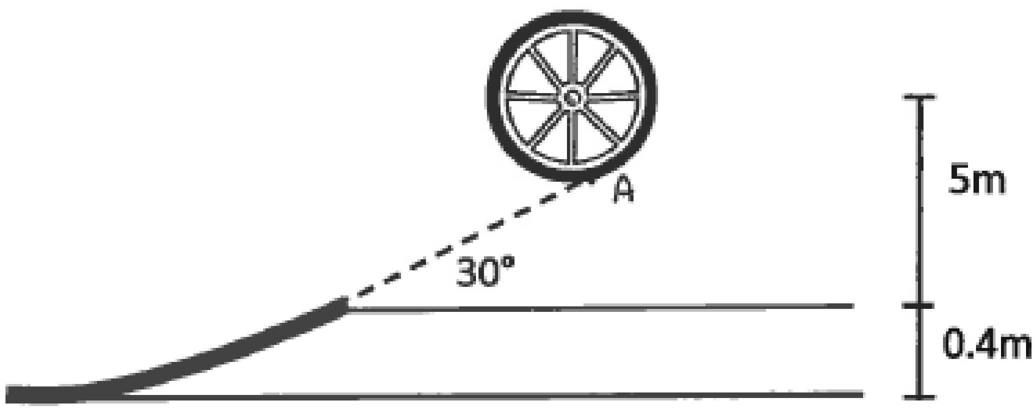
Example 1

A wheel of 4 kg mass has a pin connection at its center. A spring attached on the bottom with a cord on one end that wraps around the wheel. A couple moment of 16 N-m is acting on the wheel. Find the angle that the wheel must turn to reach an angular velocity of 8 rad/sec, starting from rest. The radius of gyration of the wheel is $k_G = 0.1$m.



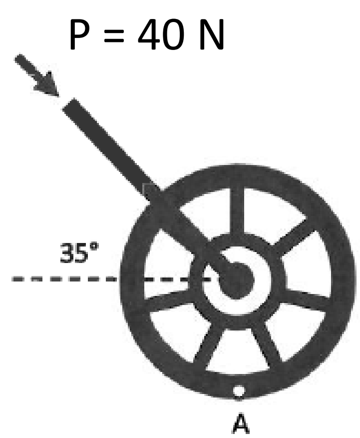
Group Problem # 27

An automobile tire has a mass of 7 kg, a radius of 0.4m, and a radius of gyration k_G =0.3m. If it is released from rest at the position shown, determine its angular velocity when it reaches the horizontal plane. The tire rolls without slipping.

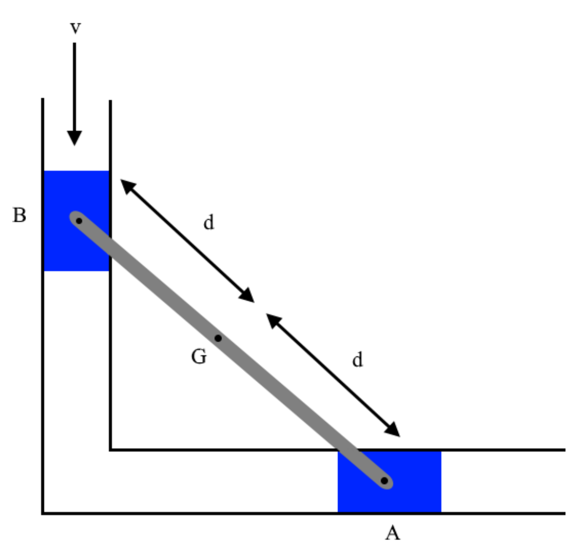


Homework Assignment # 27

1. A 40 N force is applied to the 30 kg wheel. If the wheel starts from rest and no slipping occurs, find the angular velocity ω after 5 revolutions. The radius of gyration about the center of mass is $k_G = 0.5m$.

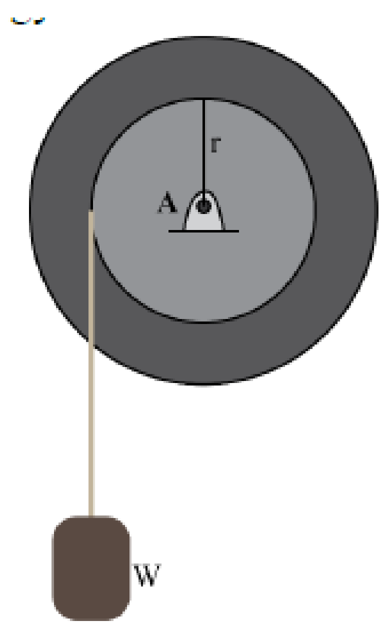


2. A 15 kg bar is restricted so that its ends move along the rectangular pathways. The bar is initially at rest when Θ = 0º and d = .3m. If the slider block B is acted on by a force of 45 N horizontally, find the angular velocity of the bar when Θ = 45º. Neglect the mass of blocks A and B as well as friction.



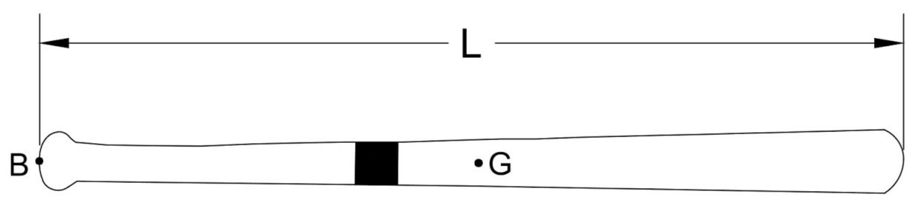
Example 2

A 20 kg block is suspended from an inextensible cable around the drum of a wheel. The drum has a radius of 0.5 m, and its moment of inertia combined with the wheel is 15N*m*s^2. The initial velocity is 2m/s downward, and a clockwise couple moment of 50 N-m is acting on the wheel. Find the velocity of the block after it moves 1.5m downward.



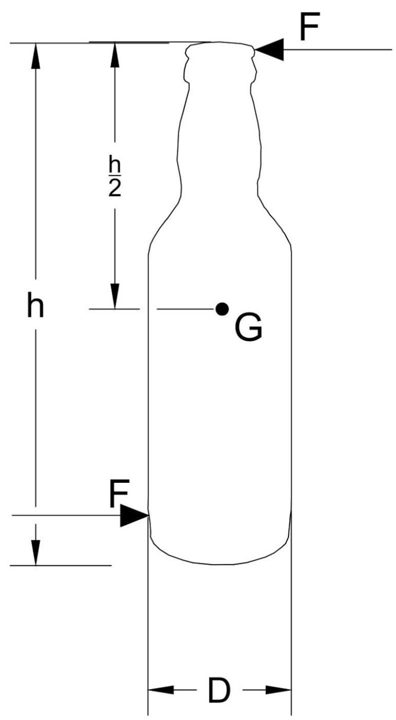
Group Problem # 27 [2]

Billy is holding a baseball bat at one end (point B on the diagram). The bat is initially horizontal and rotates about point B as it swings down because of gravity. Find the final angular velocity, ω_f, of the bat when it is vertical, with point B on the top, if its length is L = 1 m and its radius of gyration is k_B = 0.3 m (assume the bat's center of gravity, G, is at the bat's geometric center).



Homework Assignment # 27 [2]

1. Susie spins the bottle horizontally on a table by applying two equal and opposite forces on it, one on each end, as seen in the diagram, with each force being F = 0.5 lbs. She applies these forces so that they're always perpendicular to the bottle's centerline. Find the final angular velocity, ω_f, of the bottle if she only applies these forces for $\theta = 90°$. Assume the bottle to be of standard size (D = 2.5 in., h = 9 in., W = 7 oz), that the bottle can be approximated as a uniformly dense solid cylinder, and that it starts at rest.



Homework Assignment # 27 [2]

2. A tube, modeled as a solid disc of radius $\boldsymbol{\ell}$ rolls down an inclined plane inclined at angle θ. If the center of mass G is l/3 above the center of the tube at the top of the ramp, and shifts to l/3 below the center of the disc after 2.5 rotations down the ramp, determine the velocity of the disk at the final position. Assume that the initial velocity of the disc is zero. (Hint: use work and energy after finding the distance rolled and the distance "dropped" by the tube's center of mass)

The aim of this dynamics text is 1) to develop an understanding of the basic principles governing the response of bodies to forces, 2) to develop an ability to solve problems simply and logically, and 3) to apply these basic principles to practical engineering problems.

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